Monoids, Monoidal Grothendieck Construction and Clifford Semigroups

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Caviglia, Faul, Manuell and Mesiti, Clifford semigroups and the monoidal Grothendieck construction, arXiv:2607.12944, 2026.

Plan of the talk

- Variations on the monoidal Grothendieck construction
- Structure theorem for Clifford monoids
- The global category of Clifford monoids
- Application to factorization systems
- Monoids in the monoidal Grothendieck construction
- Application to inverse semirings

Discrete monoidal Grothendieck construction

The monoidal Grothendieck construction has been introduced by Joe Moeller and Christina Vasilakopoulou.

Theorem (discrete monoidal Grothendieck construction).

There are equivalences of categories

$$\begin{tikzcd}[column sep=tiny]
 & \mathcal{DFib} \simeq \mathcal{ISet} \arrow[dl, mapsto, "\text{Mon}(-)"'] \arrow[dr, mapsto, "\textit{fix base } \mathcal{X}"] & \\
\text{Mon}(\mathcal{DFib}) \simeq \text{Mon}(\mathcal{ISet}) \arrow[d, mapsto, "\textit{fix base } \mathcal{X}"'] & & \mathcal{DFib}_{\mathcal{X}} \simeq \mathcal{Cat}\,(\mathcal{X}^{\text{op}}, \mathcal{Set}) \arrow[d, mapsto, "\text{Mon}(-)"] \\
\text{Mon}(\mathcal{DFib})_{\mathcal{X}} \simeq \mathcal{MonCat}_{\text{lax}}\,(\mathcal{X}^{\text{op}}, \mathcal{Set}) & & \text{Mon}(\mathcal{DFib}_{\mathcal{X}}) \simeq \mathcal{Cat}\left(\mathcal{X}^{\text{op}}, \mathcal{Mon}\right)
\end{tikzcd}$$

This is given by the fact that **Mon(−): $\mathcal{MonCat} \to \mathcal{Cat}$ is a 2-functor**.

When $\mathcal{X}$ is strict cartesian monoidal, the two faces coincide.

Discrete monoidal Grothendieck construction

The left face.

An object of $\mathsf{Mon}(\mathcal{DFib})$ is a discrete fibration $P\colon \mathcal{E} \to \mathcal{B}$ equipped with **global multiplication $\otimes_P = (\otimes_{\mathcal{E}}, \otimes_{\mathcal{B}})$ and unit $I_P = (I_{\mathcal{E}}, I_{\mathcal{B}})$**

$$\begin{tikzcd} \mathcal{E} \times \mathcal{E} \arrow[r, "\otimes_{\mathcal{E}}"] \arrow[d, "P\times P"'] & \mathcal{E} \arrow[d, "P"] \\ \mathcal{B} \times \mathcal{B} \arrow[r, "\otimes_{\mathcal{B}}"'] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "I_{\mathcal{E}}"] \arrow[d, "\mathrm{id}"'] & \mathcal{E} \arrow[d, "P"] \\ 1 \arrow[r, "I_{\mathcal{B}}"'] & \mathcal{B} \end{tikzcd}$$

These form strict monoidal structures $(\mathcal{E}, \otimes_{\mathcal{E}}, I_{\mathcal{E}})$ and $(\mathcal{B}, \otimes_{\mathcal{B}}, I_{\mathcal{B}})$ that make P into a strict monoidal functor.

Discrete monoidal Grothendieck construction

The right face.

An object of $\mathrm{Mon}(\mathcal{DF}ib_{\mathcal{X}})$ is a discrete fibration $P : \mathcal{E} \to \mathcal{X}$ equipped with **fibrewise multiplication $\otimes$: $\mathcal{E} \times_{\mathcal{X}} \mathcal{E} \to \mathcal{E}$ and unit I: $\mathcal{X} \to \mathcal{E}$** as in the diagrams below.

$$\begin{tikzcd} \mathcal{E} \times_{\mathcal{X}} \mathcal{E} \arrow[r, "\otimes"] \arrow[d, "P \boxtimes P"'] & \mathcal{E} \arrow[d, "P"] \\ \mathcal{X} \arrow[r, equal] & \mathcal{X} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X} \arrow[r, "I"] \arrow[d, equal] & \mathcal{E} \arrow[d, "P"] \\ \mathcal{X} \arrow[r, equal] & \mathcal{X} \end{tikzcd}$$

The monoid axioms yield a monoid structure $(P_X, \otimes_X, I_X)$ on the fiber of P over any X. And reindexing functors are homomorphisms.

So the presheaf that takes fibres then factors through $\mathcal{M}on$.

Theorem (CFMM).

There are equivalences of categories

$$\begin{tikzcd}
& \mathcal{DFib} \simeq \mathcal{ISet} \arrow[dl, mapsto, "\mathrm{Grp}(-)"'] \arrow[dr, mapsto, "\textit{fix base } \mathcal{X}"] & \\
\mathrm{Grp}(\mathcal{DFib}) \simeq \mathrm{Grp}(\mathcal{ISet}) \arrow[d, mapsto, "\textit{fix base } \mathcal{X}"'] & & \mathcal{DFib}_{\mathcal{X}} \simeq \mathcal{Cat}\left(\mathcal{X}^{\mathrm{op}}, \mathcal{Set}\right) \arrow[d, mapsto, "\mathrm{Grp}(-)"] \\
\mathrm{Grp}(\mathcal{DFib})_{\mathcal{X}} \simeq \mathrm{Grp}(\mathcal{ISet})_{\mathcal{X}} & & \mathrm{Grp}(\mathcal{DFib}_{\mathcal{X}}) \simeq \mathcal{Cat}\left(\mathcal{X}^{\mathrm{op}}, \mathcal{Grp}\right)
\end{tikzcd}$$

Given by the fact that **$\mathrm{Grp}(-)\colon \mathcal{CartMonCat} \to \mathcal{Cat}$ is a 2-functor** and the equivalence **$\mathcal{DFib} \simeq \mathcal{ISet}$ is cartesian monoidal**.

Variations on the monoidal Grothendieck construction

Theorem (CFMM).

There are equivalences of categories

$$\begin{tikzcd}[column sep=tiny]
& \mathcal{DFib} \simeq \mathcal{ISet} \arrow[dl, mapsto, "\mathrm{Ab}(-)"'] \arrow[dr, mapsto, "\textit{fix base } \mathcal{X}"] & \\
\mathrm{Ab}(\mathcal{DFib}) \simeq \mathrm{Ab}(\mathcal{ISet}) \arrow[d, mapsto, "\textit{fix base } \mathcal{X}"'] & & \mathcal{DFib}_{\mathcal{X}} \simeq \mathcal{Cat}\,(\mathcal{X}^{\mathrm{op}}, \mathcal{Set}) \arrow[d, mapsto, "\mathrm{Ab}(-)"] \\
\mathrm{Ab}(\mathcal{DFib})_{\mathcal{X}} \simeq \mathrm{Ab}(\mathcal{ISet})_{\mathcal{X}} & & \mathrm{Ab}(\mathcal{DFib}_{\mathcal{X}}) \simeq \mathcal{Cat}\left(\mathcal{X}^{\mathrm{op}}, \mathcal{Ab}\right)
\end{tikzcd}$$

Given by the fact that **$\mathrm{Ab}(-)\colon \mathcal{CartMonCat} \to \mathcal{Cat}$ is a 2-functor** and the equivalence **$\mathcal{DFib} \simeq \mathcal{ISet}$ is cartesian monoidal**.

Monoidal Grothendieck construction

We will also need the usual monoidal Grothendieck construction.

Theorem (Moeller and Vasilakopoulou).

$$\begin{tikzcd}[column sep=tiny]
 & \mathcal{F}ib \simeq \mathcal{I}\mathcal{C}at \arrow[dl, mapsto, "\mathrm{PsMon}(-)"'] \arrow[dr, mapsto, "\text{fix base } \mathcal{X}"] & \\
\mathrm{PsMon}(\mathcal{F}ib) \simeq \mathrm{PsMon}(\mathcal{I}\mathcal{C}at) \arrow[d, mapsto, "\text{fix base } \mathcal{X}"'] & & \mathcal{F}ib_{\mathcal{X}} \simeq 2\mathcal{C}at_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}at) \arrow[d, mapsto, "\mathrm{PsMon}(-)"] \\
\mathrm{PsMon}(\mathcal{F}ib)_{\mathcal{X}} \simeq \mathcal{M}on2\mathcal{C}at_{\mathrm{ps}}(\mathcal{X}^{\mathrm{op}}, \mathcal{C}at) & & \mathrm{PsMon}(\mathcal{F}ib_{\mathcal{X}}) \simeq 2\mathcal{C}at_{\mathrm{ps}}\left(\mathcal{X}^{\mathrm{op}}, \mathcal{M}on\mathcal{C}at\right)
\end{tikzcd}$$

Clifford monoids

Definition.

An **inverse monoid** is a monoid M in which every element x has a unique **"weak inverse" x^*** in the sense that

- $\boldsymbol{xx^*x = x}$;
- $\boldsymbol{x^*xx^* = x^*}$.

Note that xx^* is an idempotent and $(-)^*\colon M \to M$ is an involution.

Definition.

A **Clifford monoid** is an inverse monoid in which either of the following equivalent conditions hold:

(i) $xx^* = x^*x$;

(ii) $\boldsymbol{xx^*}$ **commutes with every element** of M;

(iii) all idempotents are central.

Capturing the structure theorem for Clifford monoids

$\mathcal{Cliff}$: category of Clifford monoids and monoid homomorphisms.

$\mathcal{CInvMon}$: category of commutative inverse monoids.

$\mathcal{Cliff}_L$: subcat of $\mathcal{Cliff}$ on the monoids whose lattice of idempotents is L and the homomorphisms that fix L. Similarly $\mathcal{CInvMon}_L$.

Theorem (CFMM).

Let L be a meet-semilattice. Then the discrete monoidal Grothendieck construction ***captures, locally, every Clifford monoid in*** $\mathcal{Cliff}_L$.

$$\mathcal{Cliff}_L \simeq \mathrm{Grp}(\mathcal{DFib}_L) \simeq \mathcal{Cat}\left(L^{\mathrm{op}}, \mathcal{Grp}\right)$$

$$\mathcal{CInvMon}_L \simeq \mathrm{Ab}(\mathcal{DFib}_L) \simeq \mathcal{Cat}\left(L^{\mathrm{op}}, \mathcal{Ab}\right)$$

Capturing the structure theorem for Clifford monoids

(Idea of the proof.) Every Clifford monoid $M \in \mathcal{C}\mathit{liff}_L$ gives rise to

$$\begin{array}{rcl} L^{\mathrm{op}} & \longrightarrow & \mathcal{G}\mathit{rp} \\ e & \mapsto & G_e := \{x \in M \mid xx^* = e\} \end{array}$$

Conversely, given $C\colon L^{\mathrm{op}} \to \mathcal{G}\mathit{rp}$, we have that $\int C$ is a poset. **Since L is strict cartesian monoidal, the two faces of the discrete monoidal Grothendieck construction coincide**. So $\int C$ has an induced strict monoidal structure.

$$(e, x)^* = (e, \mathrm{inv}_{C(e)}(x)).$$

The global category of Clifford monoids

Proposition.

The functor $\mathbf{E}\colon \mathcal{Cliff} \to \mathcal{SLat}$ *that sends a Clifford monoid to its lattice of idempotents* ***is both left and right adjoint to the forgetful*** *functor.*

Corollary.

The functor $\mathbf{E}$ ***is both a fibration and an opfibration****. Explicitly, a cartesian lift of* $f\colon L \to E(B)$ *is given by the pullback*

$$\begin{tikzcd} L \times_{E(B)} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\eta_B"] \\ L \arrow[r, "f"'] & E(B) \end{tikzcd}$$

A morphism is ***cartesian if and only if it restricts to a group isomorphism*** $G_e \to G_{f(e)}$ *for each idempotent* $e \in E(M)$.

The global category of Clifford monoids

Theorem (CFMM).

$E\colon \mathcal{C}\ell iff \to \mathcal{SL}at$ is equivalent to the Grothendieck construction of

$$\begin{array}{rcl} \mathcal{C}\ell iff_{(-)}\colon \; \mathcal{SL}at^{\mathrm{op}} & \longrightarrow & \mathcal{C}at \\ L & \mapsto & \mathcal{C}\ell iff_L \simeq \mathcal{C}at\left(L^{\mathrm{op}}, \mathcal{G}rp\right) \end{array}$$

$$\begin{array}{ccc} L & & \mathcal{C}at\left(L^{\mathrm{op}}, \mathcal{G}rp\right) \\ \uparrow F & \mapsto & \downarrow - \circ F^{\mathrm{op}} \\ L' & & \mathcal{C}at\left(L'^{\mathrm{op}}, \mathcal{G}rp\right) \end{array}$$

$- \circ F^{\mathrm{op}}$ has a left adjoint, given by left Kan extension.

The global category of commutative inverse monoids

Theorem (CFMM).

$E \colon \mathcal{CInvMon} \to \mathcal{SLat}$ is equivalent to the Grothendieck construction of

$$\begin{array}{rcl} \mathcal{CInvMon}_{(-)} : \; \mathcal{SLat} & \longrightarrow & \mathcal{Cat} \\ L & \mapsto & \mathcal{CInvMon}_L \simeq \mathcal{Cat}\left(L^{\mathrm{op}}, \mathcal{Ab}\right) \end{array}$$

$$\begin{array}{ccc} L & & \mathcal{Cat}\left(L^{\mathrm{op}}, \mathcal{Ab}\right) \\ \big\downarrow F & \mapsto & \big\downarrow \mathrm{Lan}_{F^{\mathrm{op}}} \\ L' & & \mathcal{Cat}\left(L'^{\mathrm{op}}, \mathcal{Ab}\right) \end{array}$$

*Moreover, $E \colon \mathcal{CInvMon} \to \mathcal{SLat}$ is equivalent to a **monoidal opfibration**, and via the monoidal Grothendieck construction $\mathcal{CInvMon}_{(-)}$ extends to a (weakly) **lax monoidal pseudofunctor**.*

The monoidal structure of $E\colon \mathcal{CInvMon} \to \mathcal{SLat}$

The global tensor product in $\mathcal{CInvMon}$ is the one that linearizes bilinear maps (like in semirings).

This corresponds to functors

$$\tau_{L,L'}\colon \mathcal{Cat}\left(L^{\mathrm{op}}, \mathcal{Ab}\right) \times \mathcal{Cat}\left(L'^{\mathrm{op}}, \mathcal{Ab}\right) \to \mathcal{Cat}\left((L \otimes L')^{\mathrm{op}}, \mathcal{Ab}\right)$$

constructed as follows:

$$\begin{tikzcd}
L^{\mathrm{op}} \times L'^{\mathrm{op}} \arrow[r, "F\times G"] \arrow[d] & \mathcal{Ab} \times \mathcal{Ab} \arrow[d, "\otimes_{\mathcal{Ab}}"] \\
L^{\mathrm{op}} \otimes L'^{\mathrm{op}} \arrow[r, dashed, "\tau_{L,L'}(F,G)"'] & \mathcal{Ab}
\end{tikzcd}$$

Application to factorization systems

We can lift factorization systems along the bifibration $E\colon \mathcal{Cliff} \to \mathcal{SLat}$.

Lifting $(\mathbf{RegEpi}, \mathbf{Mono})$ on $\mathcal{SLat}$ along the opfibration structure we get a factorization system on $\mathcal{Cliff}$ where

- the right class consists of the morphisms which are injective on idempotents, i.e. the ***idempotent-separating morphisms***;
- the left class consists of the quotients whose kernel equivalence relations are generated by pairs of idempotents.

Can we capture inverse semirings?

We managed to capture each commutative inverse monoid, locally, as a discrete monoidal Grothendieck construction.

Then we captured the global category of commutative inverse monoids as a monoidal Grothendieck construction.

What about inverse semirings? They are monoids in the monoidal category $\mathcal{CInvMon}$, with respect to the global tensor product showed earlier.

Monoids in the monoidal Grothendieck construction

Proposition.

Let $\mathcal{A}$ be a monoidal category. ***Every (weakly) lax monoidal*** *pseudofunctor $F\colon \mathcal{A} \to \mathcal{C}at$ sends pseudomonoids to pseudomonoids. So it* ***induces*** *a pseudofunctor*

$$\widehat{F}\colon \mathrm{Mon}(\mathcal{A}) \to \mathcal{M}on\mathcal{C}at$$

Theorem (CFMM).

Let $P\colon \mathcal{E} \to \mathcal{A}$ be a monoidal opfibration, given by the monoidal Grothendieck construction of a weakly lax monoidal pseudofunctor $F\colon \mathcal{A} \to \mathcal{C}at$. Then $\mathrm{Mon}(P)\colon \mathrm{Mon}(\mathcal{E}) \to \mathrm{Mon}(\mathcal{A})$ ***is the Grothendieck construction of***

$$\mathrm{Mon}(\mathcal{A}) \xrightarrow{\widehat{F}} \mathcal{M}on\mathcal{C}at \xrightarrow{\mathrm{Mon}(-)} \mathcal{C}at$$

Application to inverse semirings

We proved that $E\colon \mathcal{CInvMon} \to \mathcal{SLat}$ is equivalent to $\int \mathcal{CInvMon}_{(-)}$

$$\begin{aligned} \mathcal{CInvMon}_{(-)}: \ \mathcal{SLat} &\longrightarrow \mathcal{Cat} \\ L &\mapsto \mathcal{CInvMon}_L \simeq \mathrm{Ab}(\mathcal{DFib}_L) \simeq \mathcal{Cat}\left(L^{\mathrm{op}}, \mathcal{Ab}\right) \end{aligned}$$

This induces a pseudofunctor $\widehat{\mathcal{CInvMon}}_{(-)}\colon \mathrm{Mon}(\mathcal{SLat}) \to \mathcal{MonCat}$
For every idempotent semiring (Q, m, e), the monoidal structure on $\mathcal{Cat}\left(Q^{\mathrm{op}}, \mathcal{Ab}\right)$ coincides with that of the **Day convolution**.

Theorem (CFMM).

The category of ***inverse semirings*** *is equivalent to the Grothendieck construction of* $\mathrm{Mon}(\mathcal{SLat}) \xrightarrow{\widehat{\mathcal{CInvMon}}_{(-)}} \mathcal{MonCat} \xrightarrow{\mathrm{Mon}(-)} \mathcal{Cat}$.
So an inverse semiring is an idempotent semiring Q *equipped with a* ***lax monoidal functor*** $Q^{\mathrm{op}} \to \mathcal{Ab}$.