Towards elementary 2-toposes

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1-dimensional elementary topoi

Definition.

An **elementary topos** is a category $\mathcal{C}$ that

(E1) has finite limits;

(E2) is cartesian closed;

(E3) has a **subobject classifier**.

Finite limits include products (which model logical conjunction) and pullbacks (which model logical substitution).

Cartesian closed means that for every pair $X, Y \in \mathcal{C}$ there is an object Y^X representing maps from X to Y (this models logical implication).

Subobject classifiers

The archetypal subobject classifier is given by the characteristic functions, exhibiting $\mathcal{S}et$ as **the archetypal elementary topos**.

For every set X, **there is a bijection**

$$\{\text{subsets of } X\} \cong \{\text{functions } X \to \{T, F\}\}$$

which identifies a subset A of X with the **characteristic function** of A

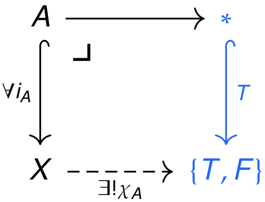
$$\begin{aligned} \chi_A : \quad X &\longrightarrow \{T, F\} \\ x &\longmapsto \begin{cases} T \text{ if } x \in A \\ F \text{ if } x \notin A \end{cases} \end{aligned}$$

Subsets and hence (logical propositions) **are classified by $\{T, F\}$**.

$\boldsymbol{T\colon * \hookrightarrow \{T, F\}}$ which picks T **is the archetypal subobject classifier**.

We can capture it with a universal property!

The subset $\boldsymbol{T\colon * \hookrightarrow \{T, F\}}$ **is universal**, in the sense that for every subset $i_A\colon A \hookrightarrow X$ in $\mathcal{S}et$, there exists a unique function $\chi_A\colon X \to \{T, F\}$, namely the characteristic function of A, such that the following is a **pullback**:



Subobject classifiers

Definition.

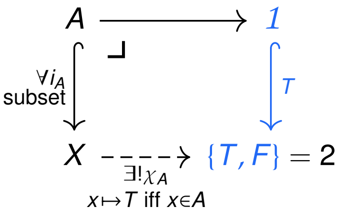
A **subobject classifier** in a category $\mathcal{E}$ is a **monomorphism** $\tau\colon * \hookrightarrow \Omega$ such that **every monomorphism** $i\colon A \hookrightarrow X$ in $\mathcal{E}$ **is the pullback of τ along a unique morphism** $\chi_i\colon X \to \Omega$, called the characteristic morphism of i:

$$\begin{array}{ccc} A & \longrightarrow & * \\ {\scriptstyle \forall i}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathsf{true}} \\ X & \underset{\exists!\chi_i}{\dashrightarrow} & \Omega \end{array}$$

Ω represents an **object of generalized truth values**.

If $\mathcal{E}$ has a subobject classifier, has all finite limits and is cartesian closed, then $\mathcal{E}$ is called an elementary topos. In particular, it has an

$\mathcal{S}et$ is the archetypal elementary topos



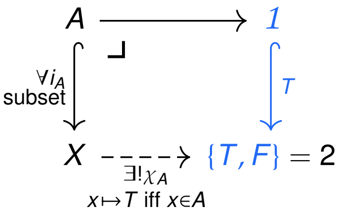
$$\forall X \in \mathcal{S}et$$

$$\mathcal{G}_{T,X} \colon \mathcal{S}et\,(X, 2) \xrightarrow[\text{pb along } T]{} \mathrm{Sub}(X)$$

is a bijection.

1-dimensional elementary topoi

$\mathcal{S}et$ is the archetypal elementary topos



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Definition.

Let $\mathcal{C}$ be a 1-category with finite limits. A **subobject classifier** in $\mathcal{C}$ is a map $\tau \colon 1 \hookrightarrow \Omega$ in $\mathcal{C}$ such that $\forall X \in \mathcal{C}$ **$\mathcal{G}_{T,X} \colon \mathcal{C}\,(X, \Omega) \to \mathrm{Sub}(X)$ given by pulling back τ is a bijection**.

Properties of elementary topoi

Many properties can be deduced from this simple definition, including: finite colimits, power objects, locally cartesian closedness, extensivity, regularity, Barr-exactness, Heyting category.

2-dimensional elementary topoi

$\mathcal{C}at$ should be the archetypal elementary 2-topos

$\mathcal{C}at$ satisfies good properties: complete, cocomplete, cartesian closed, regular, Barr-exact.

2-dimensional elementary topoi

$\mathcal{Cat}$ should be the archetypal elementary 2-topos

$\mathcal{Cat}$ satisfies good properties: complete, cocomplete, cartesian closed, regular, Barr-exact.

But other properties are not fully satisfied:

(i) we can only form categories of presheaves on a small category;

(ii) only Conduché functors are exponentiable.

What about the classifier?

2-dimensional classifier

In dimension 2, Weber proposed to classify discrete opfibrations. Their fibres are sets, thus of one dimension higher than the fibres of subsets.

Definition.

A **discrete opfibration** is a functor $p\colon \mathcal{E} \to \mathcal{B}$ such that for every $E \in \mathcal{E}$ every $f\colon p(E) \to B$ in $\mathcal{B}$ has a unique lifting to E.

$$
\begin{array}{c}
\mathcal{E} \\
\Big\downarrow p \\
B
\end{array}
\qquad\qquad
\begin{array}{ccc}
E & \overset{\exists!\, \overline{f}^{E}}{\dashrightarrow} & f_{*}E \\
\Big\downarrow p & & \Big\downarrow p \\
p(E) & \underset{f}{\longrightarrow} & B
\end{array}
$$

Discrete opfibrations in a 2-category are defined by representability.

2-dimensional classifier

$\mathcal{C}at$ should be the archetypal elementary 2-topos

Its 2-dimensional classification is given by the **category of elements (Grothendieck construction)**, that exhibits equivalences

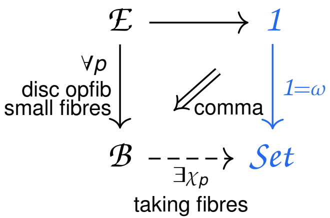
$$\mathcal{C}at\left(\mathcal{B}, \mathcal{S}et\right) \simeq \mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}\left(\mathcal{B}\right)$$

between copresheaves and discrete opfibrations with small fibres.

$\mathcal{S}et$ represents the object of generalized truth values, allowing for a more expressive internal logic.

2-dimensional classifier

There are **two equivalent ways to capture the category of elements**. It is given by the **comma object**



$\forall \mathcal{B} \in \mathcal{C}at$

$$\mathcal{C}at\left(\mathcal{B}, \mathcal{S}et\right) \xrightarrow[\text{comma along } \omega]{\widehat{\mathcal{G}}_{\omega,\mathcal{B}}} \mathcal{DO}p\mathcal{F}ib^{s}\left(\mathcal{B}\right)$$

is an equivalence of categories.

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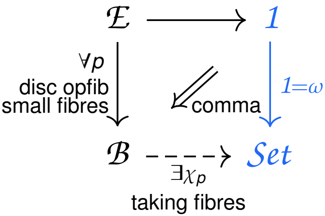

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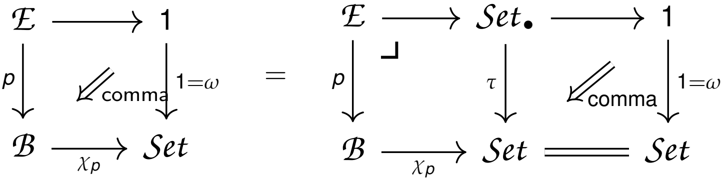
And equivalently by the **pullback along the replacement** of ω:



2-dimensional classifier

Definition (M., stronger version of Weber's notion).

Let $\mathcal{L}$ be a 2-category with comma objects, pullbacks along discrete opfibrations and terminal objects. Let P be a fixed pullback stable property for discrete opfibrations. A **good 2-classifier in $\mathcal{L}$** (w.r.t. P) is a morphism **$\omega\colon 1 \to \Omega$ in $\mathcal{L}$** such that for every $F \in \mathcal{L}$ the functor

$$\widehat{\mathcal{G}}_{\omega,F}\colon \mathcal{L}(F,\Omega) \to \mathcal{DOpFib}(F)$$

given by taking comma objects from ω forms an equivalence of categories when restricting the codomain to the full subcategory $\mathcal{DOpFib}^{\mathrm{P}}(F)$ on the discrete opfibrations that satisfy P.

Reduction of 2-classifiers to dense generators

Definition.

A 2-functor $I \colon \mathcal{Y} \underset{\text{ff}}{\hookrightarrow} \mathcal{L}$ is **dense** if the restricted Yoneda embedding

$$\begin{aligned} \widetilde{I} \colon \mathcal{L} &\longrightarrow [\mathcal{Y}^{\mathrm{op}}, \mathcal{C}at] \\ F &\mapsto \mathcal{L}(I(-), F) \end{aligned}$$

is fully faithful.

Equivalently, **every $F \in \mathcal{L}$ is an I-absolute** (i.e. preserved by $\widetilde{I}$) **2-colimit of a diagram that factors through $\mathcal{Y}$**.

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Example.

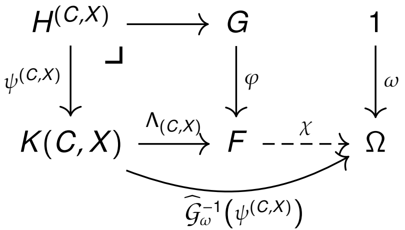
Representables form a dense generator $\mathrm{y}\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ of 2-presheaves. Every 2-presheaf is a weighted 2-colimit of representables.

Theorem (M.).

Let $I\colon \mathcal{Y} \underset{\text{ff}}{\hookrightarrow} \mathcal{L}$ *be dense and consider* $\omega\colon 1 \to \Omega$ *in* $\mathcal{L}$. ***TFAE***:

(*i*) ω *is a good 2-classifier in* $\mathcal{L}$, *i.e.* ***$\forall F \in \mathcal{L}$ $\widehat{\mathcal{G}}_{\omega,F}$ is an equivalence***;

(*ii*) ***$\forall Y \in \mathcal{Y}$ $\widehat{\mathcal{G}}_{\omega,Y}$ is an equivalence*** *and a certain* ***operation of normalization*** *is possible.*



M. 2-classifiers via dense generators and Hofmann-Streicher universe in stacks, *Canadian Journal of Mathematics*, 2024.

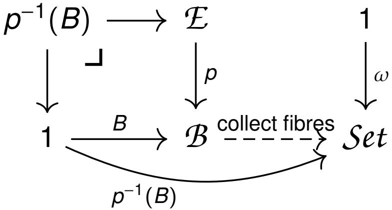
Reduction applied to $\mathcal{C}at$

Example.

The singleton category ***1* is dense in $\mathcal{C}at$**. **So we can just look at the classification over *1***, where we clearly have an equivalence.

$$\widehat{\mathcal{G}}_{\omega,1}\colon \mathcal{C}at\,(1,\mathcal{S}et) \xrightarrow{\sim} \mathcal{DO}p\mathcal{F}ib^{s}(1) \cong \mathcal{S}et$$

We deduce from this trivial observation that the category of elements construction is fully faithful and classifies precisely all discrete opfibrations with small fibres.



2-categories of stacks: Grothendieck 2-topoi

Definition (Idea).

A **stack** is a **bicategorical sheaf**. It is a 2-functor $F\colon \mathcal{C}^{\text{op}} \to \mathcal{C}at$ such that, for every $C \in \mathcal{C}$ and covering sieve $S \in J(C)$, every assignment

$$(D \xrightarrow{f} C) \in S \;\longmapsto\; M_f \in F(D)$$
$$(D' \xrightarrow{g} D \xrightarrow[\in S]{f} C) \;\longmapsto\; \varphi^{f,g}\colon g^* M_f \cong M_{f\circ g}$$

with the $\varphi^{f,g}$ satisfying the cocycle condition can be **glued into a global** $M \in F(C)$ with coherent isomorphisms $\psi^f\colon f^* M \cong M_f$.

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with the $\varphi^{f,g}$ satisfying the cocycle condition can be **glued into a global** $M \in F(C)$ with coherent isomorphisms $\psi^f\colon f^* M \cong M_f$.

Moreover F is required to be a sheaf on morphisms, i.e. we can glue matching families of morphisms $f^* X \to f^* Y$ in $F(D)$ in a unique way into global morphisms $X \to Y$ in $F(C)$.

A good 2-classifier in 2-presheaves

Let $\mathcal{C}$ be a category and consider $\mathcal{L} = [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$. **Representables form a dense generator** $\mathrm{y}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$, so we can just look at

$$\widehat{\mathcal{G}}_{\omega,\mathrm{y}(C)}\colon [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]\,(\mathrm{y}(C), \Omega) \to \mathcal{DO}p\mathcal{F}ib^{\,\mathrm{s}}(\mathrm{y}(C))$$

We want all these functors to be equivalences of categories. So the **Yoneda lemma forces a good 2-classifier** Ω to be, up to equivalence,

$$C \overset{\Omega}{\mapsto} \mathcal{DO}p\mathcal{F}ib^{\,\mathrm{s}}(\mathrm{y}(C))$$

$\omega\colon 1 \to \Omega$ picks the identity on every component.

However, this **Ω is only a pseudofunctor, and it is not clear that it lands in small categories**.

Indexed Grothendieck construction

Theorem (Caviglia–M.).

For every 2-functor $F\colon \mathcal{A} \to \mathcal{C}at$, there is a pseudonatural equivalence

$$\mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}(F) \simeq \left[\int F, \mathcal{C}at\right]$$

between split opfibrations in $\left[\mathcal{A}, \mathcal{C}at\right]$ over F and 2-copresheaves on the Grothendieck construction $\int F$ of F.

This restricts to a pseudonatural equivalence

$$\mathcal{D}\mathcal{O}p\mathcal{F}ib^{s}_{[\mathcal{A},\mathcal{C}at]}(F) \simeq \left[\int F, \mathcal{S}et\right].$$

This is a **2-dimensional generalization of the fundamental theorem of elementary topos theory**, in the Grothendieck topos case.

Caviglia and M. Indexed Grothendieck construction, *TAC*, 2024.

A good 2-classifier in 2-presheaves

Theorem (M.).

$$\begin{array}{rccc} \widetilde{\Omega}: & \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathcal{C}at \\ & C & \mapsto & \left[\left(\mathcal{C}/_{C}\right)^{\mathrm{op}}, \mathcal{S}et\right] \\ & (C \xleftarrow{f} D) & \mapsto & - \circ (f \circ =)^{\mathrm{op}}, \end{array}$$

equipped with $\widetilde{\omega}\colon 1 \to \widetilde{\Omega}$ *that picks the constant at 1 presheaf on every component,* ***is a good 2-classifier in*** $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ *that classifies all discrete opfibrations with small fibres.*

M. 2-classifiers via dense generators and Hofmann-Streicher universe in stacks, *Canadian Journal of Mathematics*, 2024.

A good 2-classifier in 2-presheaves

The proof actually involves **the bicategorical classification process of $\Omega\colon C \mapsto \mathcal{DOpFib}^{\mathrm{s}}(\mathrm{y}(C))$**

$$\widehat{\mathcal{G}}_{\omega,\mathrm{y}(C)}\colon \mathrm{Ps}\left[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}\right](\mathrm{y}(C), \Omega) \to \mathcal{DOpFib}^{\mathrm{s}}_{[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]}(\mathrm{y}(C)) = \Omega(C)$$

which **is isomorphic to the Yoneda lemma's map** and is thus an equivalence.

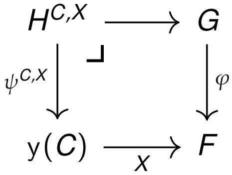
A good 2-classifier in 2-presheaves

The proof actually involves **the bicategorical classification process of Ω: $C \mapsto \mathcal{DOpFib}^{\mathrm{s}}(\mathrm{y}(C))$**

$$\widehat{\mathcal{G}}_{\omega,\mathrm{y}(C)} \colon \mathrm{Ps}\left[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}\right](\mathrm{y}(C), \Omega) \to \mathcal{DOpFib}^{\mathrm{s}}{}_{[\mathcal{C}^{\mathrm{op}},\mathcal{C}at]}(\mathrm{y}(C)) = \Omega(C)$$

which **is isomorphic to the Yoneda lemma's map** and is thus an equivalence.

Idea of the normalization process:

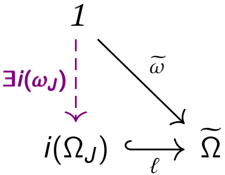


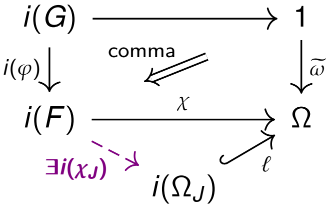
change the fibre $\left(\psi_D^{C,X}\right)_{D \xrightarrow{f} C}$ into the fibre $(\varphi_D)_{F(f)(X)}$ (fibres of φ are global).

Restricting $\widetilde{\Omega}$ to a good 2-classifier in stacks

Consider $i\colon \mathcal{S}t(\mathcal{C},J) \subseteq [\mathcal{C}^{op},\mathcal{C}at]$ the full 2-subcategory on **stacks**.
We restrict $\widetilde{\omega}\colon 1 \to \widetilde{\Omega}$ to a good 2-classifier in stacks.

We proved with a general argument that it is enough to find $\Omega_J \in \mathcal{S}t(\mathcal{C},J)$ and $\ell\colon i(\Omega_J) \underset{\text{ff}}{\hookrightarrow} \widetilde{\Omega}$ **chronic** such that, given $\varphi\colon G \to F$ a discrete opfibration in $\mathcal{S}t(\mathcal{C},J)$ with small fibres





Then $\omega_J\colon 1 \to \Omega_J$ is a good 2-classifier in $\mathcal{S}t(\mathcal{C},J)$.

A good 2-classifier in stacks

Theorem (M.).

$$\begin{array}{rcl} \Omega_J : \quad \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathcal{C}at \\ C & \mapsto & \mathcal{S}h(\mathcal{C}/C, J) \subseteq \left[\left(\mathcal{C}/C\right)^{\mathrm{op}}, \mathcal{S}et\right] \\ (C \xleftarrow{f} D) & \mapsto & -\circ (f\circ =)^{\mathrm{op}}, \end{array}$$

equipped with $\omega_J \colon 1 \to \Omega_J$ that picks the constant at 1 sheaf on every component, ***is a good 2-classifier in $\mathcal{S}t(\mathcal{C}, J)$*** *that classifies all discrete opfibrations with small fibres.*

This also solves a problem posed by Hofmann and Streicher when attempting to lift Grothendieck universes to sheaves.

M. 2-classifiers via dense generators and Hofmann-Streicher universe in stacks, *Canadian Journal of Mathematics*, 2024.

Other possible axioms of elementary 2-topos

Definition (Weber).

Let $\mathcal{K}$ be a 2-category. A duality involution for $\mathcal{K}$ is an involution 2-functor $(-)^\circ \colon \mathcal{K}^{co} \to \mathcal{K}$ equipped with pseudonatural equivalences of categories

$$\mathcal{DFib}_{\mathcal{K}}(A \times B, C) \simeq \mathcal{DFib}_{\mathcal{K}}(A, B^\circ \times C)$$

$(-)^{op} \colon \mathcal{C}at^{co} \to \mathcal{C}at$ is a duality involution for $\mathcal{C}at$.

For presheaves on a 1-category $\mathcal{C}$ (and stacks over a site $\mathcal{C}$), one can consider the pointwise opposite: the opposite of $F \colon \mathcal{C}^{op} \to \mathcal{C}at$ is

$$\mathcal{C}^{op} \cong \mathcal{C}^{coop} \xrightarrow{F^{co}} \mathcal{C}at^{co} \xrightarrow{(-)^{op}} \mathcal{C}at$$

But for presheaves on a 2-category $\mathcal{L}$ it's trickier.

Properties of Grothendieck 2-toposes

Proposition (M.).

The 2-category $\mathcal{S}t(\mathcal{C}, J)$ *of stacks* ***has all flexible limits*** (*thus all comma objects and the terminal object*) *and all pullbacks along discrete opfibrations, calculated in* $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ *and hence pointwise.*

It looks like flexible limits are the right ones to consider in the definition of elementary 2-topos.

Properties of Grothendieck 2-toposes

Proposition (M.).

The 2-category $\mathcal{S}t(\mathcal{C}, J)$ *of stacks* ***has all flexible limits*** *(thus all comma objects and the terminal object) and all pullbacks along discrete opfibrations, calculated in* $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ *and hence pointwise.*

It looks like flexible limits are the right ones to consider in the definition of elementary 2-topos.

The codomain $\Omega_J \colon C \mapsto \mathcal{S}h(\mathcal{C}/_{C}, J)$ of the good 2-classifier in $\mathcal{S}t(\mathcal{C}, J)$ **is probably an internal topos**. (In dimension 1 we have that Ω is always an internal Heyting algebra). Weber has already shown internal cartesian closedness (with some assumptions).