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joint work with Elena Caviglia

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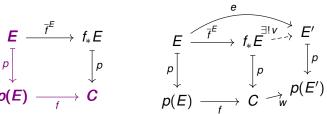


Caviglia—M. Indexed Grothendieck construction, arXiv:2307.16076.

## Grothendieck opfibrations

#### Definition.

A functor  $p: \mathcal{E} \to \mathcal{C}$  is called a **(Grothendieck) opfibration** (in  $\mathcal{CAT}$ ) (over  $\mathcal{C}$ ) if for all  $E \in \mathcal{E}$  and  $f: p(E) \to C$  in  $\mathcal{C}$ , there exists an opcartesian lifting  $\overline{f}^E: E \to f_*E$  of f to E



We call **cleavage** a choice of the  $\overline{f}^E$ . We call p **split** if  $\overline{id}^E = id_E$  and  $\overline{g}^{f_*E} \circ \overline{f}^E = \overline{(g \circ f)}^E$ . We call p **discrete** if the liftings  $\overline{f}^E$  are unique.

#### Grothendieck construction

#### Construction.

Let  $\mathcal C$  be a category and let  $F\colon \mathcal C\to\mathcal C\!\mathcal A\mathcal T$  be a 2-functor.

The **Grothendieck construction** of F is the functor  $\mathcal{G}(F): \int F \to \mathcal{C}$  of projection on the first component from the category  $\int F$ .

An object of  $\int F$  is a pair (C, X) with  $C \in C$  and  $X \in F(C)$ ; a morphism  $(C, X) \to (D, X')$  in  $\int F$  is a pair  $(f, \alpha)$  with  $f: C \to D$  a morphism in C and  $\alpha: F(f)(X) \to X'$  a morphism in F(D).

 $\mathcal{G}(F): \int F \to \mathcal{C}$  is a split opfibration, with cleavage given by the morphisms  $(f, \mathrm{id})$ . Every morphism  $(f, \alpha)$  can be factorized as

$$(C,X) \xrightarrow{(f,id)} (D,F(f)(X)) \xrightarrow{(id,\alpha)} (D,X')$$



## Grothendieck construction, abstractly

$$\int F = \text{oplax-colim} F$$

$$\operatorname{inc}_C: F(C) \longrightarrow \int F$$

$$\begin{array}{ccc}
X & (C, X) \\
\downarrow^{\alpha} & \mapsto & \downarrow^{(\operatorname{id}, \alpha)} \\
X' & (C, X')
\end{array}$$

$$F(C) \xrightarrow{\operatorname{inc}_{C}} \int F$$

$$F(f) \downarrow \operatorname{inc}_{f} \downarrow$$

$$F(D) \xrightarrow{\operatorname{inc}_{D}}$$

$$(\operatorname{inc}_{f})_{X} = (f, \operatorname{id})$$

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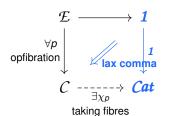
$$\begin{matrix} X & (C,X) \\ \downarrow^{\alpha} & \mapsto & \downarrow^{(\operatorname{id},\alpha)} \\ X' & (C,X') \end{matrix}$$

$$F(C) \xrightarrow{\operatorname{inc}_{C}} \int F$$

$$F(f) \downarrow \qquad \operatorname{inc}_{D}$$

$$F(D) \qquad \operatorname{inc}_{D}$$

$$(\operatorname{inc}_{f})_{X} = (f, \operatorname{id})$$



#### equivalence of categories

$$[C, CAT] \simeq OpFib(C)$$
$$[C, Set] \simeq DOpFib(C)$$

# Opfibrations in a 2-category *L*

#### Definition.

A split optibration in  $\mathcal{L}$  is  $\varphi \colon G \to F$  in  $\mathcal{L}$  s.t. for every  $X \in \mathcal{L}$ 

$$\varphi \circ -: \mathcal{L}(X, G) \to \mathcal{L}(X, F)$$

is a split optibration (in CAT) and for every  $\lambda \colon K \to X$  in L

$$\begin{array}{ccc}
\mathcal{L}(X,G) \stackrel{-\circ\lambda}{\to} \mathcal{L}(K,G) \\
\varphi \circ -\downarrow & \downarrow \varphi \circ -\\
\mathcal{L}(X,F) \xrightarrow{-\circ} \mathcal{L}(K,F)
\end{array}$$

is cleavage preserving.

# Opfibrations in a 2-category $\mathcal{L}$

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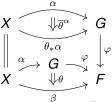
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\varphi\circ -\downarrow & & \downarrow \varphi\circ -\\
\mathcal{L}(X,F) \xrightarrow[-\circ\lambda]{} \mathcal{L}(K,F)
\end{array}$$

is cleavage preserving.

In CAT, it suffices to look at X = 1



# Opfibrations in [A, CAT]

### Proposition (Caviglia-M.).

Let  $\mathcal{A}$  be a category. The following facts are equivalent:

- (i)  $\varphi : G \to F$  is a split optibration in [A, CAT];
- (ii) every component  $\varphi_A \colon G(A) \to F(A)$  is a split optibration (in  $\mathcal{CAT}$ ) and for every  $h \colon A \to B$  in  $\mathcal{A}$  the naturality square

$$G(A) \xrightarrow{G(h)} G(B)$$
 $\varphi_A \downarrow \qquad \qquad \downarrow \varphi_B$ 
 $F(A) \xrightarrow{F(h)} F(B)$ 

is cleavage preserving.

Idea: it suffices to look at the representables.

#### Theorem (Caviglia-M.).

For every 2-functor  $F: \mathcal{A} \to \mathcal{CAT}$ , there is an **equivalence of** categories

$$OpFib_{[\mathcal{A},C\mathcal{A}T]}(F)\simeq\left[\int F,C\mathcal{A}T\right]$$

This restricts to an equivalence of categories

$$\mathcal{D}Op\mathcal{F}ib_{[\mathcal{A},\mathcal{CAT}]}(F)\simeq\left[\int F,\mathcal{S}et\right]$$

Moreover, both the equivalences are pseudonatural in F.

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Moreover, both the equivalences are pseudonatural in F.

When  $\mathcal{A}=1$ , we recover the usual Grothendieck construction. Indeed  $F: 1 \to \mathcal{CAT}$  is just a category  $\mathcal{C}$  and  $\int F = \mathcal{C}$ .

$$OpFib(C) \simeq [C, CAT].$$

$$\mathcal{D}Op\mathcal{F}ib_{[\mathcal{A},\mathcal{CAT}]}(F)\simeq\left[\int F,\mathcal{S}et
ight]$$

further reduces, when  $F: \mathcal{A} \to Set$ , to the well-known

$$[\mathcal{A}, \mathcal{S}et]/_{\mathcal{F}} \simeq \left[\int \mathcal{F}, \mathcal{S}et\right].$$

When F is a representable  $y(A) : A \rightarrow Set$ , this is the famous

$$[\mathcal{A}, Set]/y(A) \simeq [\mathcal{A}/A, Set].$$

We find the Grothendieck topoi case of the **fundamental theorem of elementary topos theory**.

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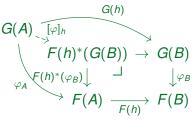
We find the Grothendieck topoi case of the **fundamental theorem of elementary topos theory**. Our equivalence

$$\mathit{OpFib}_{[\mathcal{A},\mathcal{CAT}]}(\mathit{F})\simeq\left[\int\!\!\mathit{F},\mathcal{CAT}
ight]$$

shows that every (op)fibrational slice of a Grothendieck 2-topos is again a Grothendieck 2-topos.

Idea of the proof:

$$\begin{split} \left[\int F, \mathcal{C}at\right] & \cong \left[\mathcal{A}^{\mathrm{op}}, \mathcal{C}\mathcal{A}T\right]_{\mathrm{oplax}} \left(\Delta 1, \left[F(-), \mathcal{C}at\right]\right) & \simeq \\ & \cong \operatorname{Ps}\left[\mathcal{A}^{\mathrm{op}}, \mathcal{C}\mathcal{A}T\right]_{\mathrm{oplax}} \left(\Delta 1, \mathcal{O}p\mathcal{F}ib_{\mathcal{C}\mathcal{A}T}(F(-))\right) & \cong \mathcal{O}p\mathcal{F}ib_{\left[\mathcal{A}, \mathcal{C}\mathcal{A}T\right]}(F) \\ & \int F = \operatorname{oplax-colim} F \\ & \mathcal{G}_{-} \colon \left[-, \mathcal{C}at\right] \simeq \mathcal{O}p\mathcal{F}ib\left(-\right) \end{split}$$



## **Explicit indexed Grothendieck construction**

An opfibration  $\varphi \colon G \to F$  in  $[\mathcal{A}, \mathcal{CAT}]$  is sent to

$$\int F \longrightarrow \mathcal{CAT}$$

$$(A, X) \qquad (\varphi_A)_X \\
\downarrow^{(h,id)} \qquad \downarrow^{G(h)}$$

$$(B, F(f)(X)) \mapsto (\varphi_B)_{F(h)(X)} \\
\downarrow^{(id,\alpha)} \qquad \downarrow^{\alpha_*} \\
(B, X') \qquad (\varphi_B)_{X'}$$

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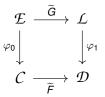
$$\downarrow (id, \alpha) \qquad \qquad \downarrow \alpha_*$$

$$(B, X') \qquad (\varphi_B)_{X'}$$

A 2-functor  $Z: \int F \to \mathcal{CAT}$  is sent to the opfibration whose component  $G(A) \to F(A)$  is the projection on the first component with **an object of** G(A) is a **pair**  $(X,\xi)$  with  $X \in F(A)$  and  $\xi \in Z(A,X)$ ; **a morphism**  $(X,\xi) \to (X',\xi')$  in G(A) is a **pair**  $(\alpha,\Lambda)$  with  $\alpha: X \to X'$  in F(A) and  $\Lambda: Z(\operatorname{id},\alpha)(\xi) \to \xi'$  in Z(A,X').

### Example ( $\mathcal{A}=2$ ).

When  $\mathcal{A} = 2$ , we have  $[\mathcal{A}, \mathcal{CAT}] = \mathcal{CAT}^2$ , so that  $F: 2 \to \mathcal{CAT}$  is a functor  $\widetilde{F}: \mathcal{C} \to \mathcal{D}$ . A split optibration  $\varphi: G \to F$  in  $[2, \mathcal{CAT}]$  is a cleavage preserving morphism between split optibrations in  $\mathcal{CAT}$ 



This can be **reorganized as a 2-functor**  $\int F \to \mathcal{CAT}$ . The objects of  $\int F$  are the disjoint union of the objects of  $\mathcal C$  and of  $\mathcal D$ . The morphisms in  $\int F$  are of three kinds: morphisms in  $\mathcal C$  (over 0), morphisms in  $\mathcal D$  (over 1) and morphisms over  $0 \to 1$  that represents the objects  $(C, D, \widetilde F(C) \to D)$  of the comma category  $\widetilde F/\mathcal D$ .

## Example ( $\mathcal{A} = \Delta$ ).

When  $\mathcal A$  is the simplex category  $\Delta$ , we have that  $F:\Delta\to\mathcal{CAT}$  is a cosimplicial category. Split optibrations between cosimplicial categories over F are equivalent to 2-copresheaves on the total category that collects all the cosimplexes given by F.

## Example (semidirect product of groups).

Let  $\mathcal{A}$  be the one-object category  $\mathcal{BG}$  corresponding to a group  $\mathcal{G}$ . A functor  $F \colon \mathcal{BG} \to \mathcal{CAT}$  which picks some  $\mathcal{BH}$  corresponds with  $\rho \colon \mathcal{G} \to \operatorname{Aut}(\mathcal{H})$ . Then  $\int F = \mathcal{B}(\mathcal{H} \rtimes_{\rho} \mathcal{G})$ . Functors  $\mathcal{B}(\mathcal{H} \rtimes_{\rho} \mathcal{G}) \to \mathcal{CAT}$  are equivalent to optibrations in  $[\mathcal{BG}, \mathcal{CAT}]$  over the functor F that corresponds with  $\rho \colon \mathcal{G} \to \operatorname{Aut}(\mathcal{H})$ .

#### Example.

**2-classifiers**, introduced by Weber, are the generalization to dimension 2 of subobject classifiers. *CAT* is the archetypal elementary **2-topos**. Its 2-dimensional classification process is the Grothendieck construction.

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The indexed Grothendieck construction is a key element to show a 2-classifier in 2-presheaves.

$$\Omega(A) = \mathcal{D}Op\mathcal{F}ib_{[\mathcal{A}^{op},\mathcal{CAT}]}(y(A)) \simeq [(\mathcal{A}/_{A})^{op},\mathcal{S}et] = \widetilde{\Omega}(A)$$

 $\widetilde{\Omega}$  is a strictly functorial replacement of  $\Omega.$ 

### Theorem (M.).

 $\tilde{\Omega}$  is a 2-classifier in [ $\mathcal{A}^{op}$ ,  $\mathcal{CAT}$ ] that classifies all discrete optibrations with small fibres. It can be also restricted to a 2-classifier in stacks.