Indexed Grothendieck construction

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joint work with Elena Caviglia

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Caviglia–M. Indexed Grothendieck construction, *arXiv:2307.16076*.

Grothendieck opfibrations

Definition.

A functor $p\colon \mathcal{E} \to \mathcal{C}$ is called a **(Grothendieck) opfibration** (in $\mathcal{CAT}$) (over $\mathcal{C}$) if for all **$E \in \mathcal{E}$** and **$f\colon p(E) \to C$ in $\mathcal{C}$**, there exists an **opcartesian lifting $\overline{f}^E\colon E \to f_*E$ of f to E**

$$\begin{tikzcd}
E \arrow[r, "\overline{f}^E"] \arrow[d, mapsto, "p"'] & f_*E \arrow[d, mapsto, "p"] \\
p(E) \arrow[r, "f"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\overline{f}^E"] \arrow[rr, bend left, "e"] \arrow[d, mapsto, "p"'] & f_*E \arrow[r, dashed, "\exists!\, v"] \arrow[d, mapsto, "p"] & E' \arrow[d, mapsto, "p"] \\
p(E) \arrow[r, "f"'] & C \arrow[r, "w"'] & p(E')
\end{tikzcd}$$

We call **cleavage** a choice of the $\overline{f}^E$. We call p **split** if $\overline{\mathrm{id}}^E = \mathrm{id}_E$ and $\overline{g}^{f_*E} \circ \overline{f}^E = \overline{(g \circ f)}^E$. We call p **discrete** if the liftings $\overline{f}^E$ are unique.

Grothendieck construction

Construction.

Let $\mathcal{C}$ be a category and let $F\colon \mathcal{C} \to \mathcal{CAT}$ be a 2-functor.
The **Grothendieck construction** of F is the functor $\mathcal{G}(F)\colon \int F \to \mathcal{C}$ of projection on the first component from the category $\int F$.

An object of $\int F$ is a **pair (C, X)** with $C \in \mathcal{C}$ and $X \in F(C)$;

a morphism $(C, X) \to (D, X')$ in $\int F$ is a **pair (f, α)** with $f\colon C \to D$ a morphism in $\mathcal{C}$ and $\alpha\colon F(f)(X) \to X'$ a morphism in $F(D)$.

$\mathcal{G}(F)\colon \int F \to \mathcal{C}$ is a split opfibration, with cleavage given by the morphisms (f, id). Every morphism (f, α) can be factorized as

$$(C, X) \xrightarrow{(f,\mathrm{id})} (D, F(f)(X)) \xrightarrow{(\mathrm{id},\alpha)} (D, X')$$

Grothendieck construction, abstractly

$$\int F = \operatorname{oplax-colim} F$$

$$\begin{tikzcd}[row sep=small]
\mathrm{inc}_C : & F(C) \arrow[r] & \int F \\
& X \arrow[d, "\alpha"] & (C, X) \arrow[d, "(\mathrm{id}, \alpha)"] \\
& X' & (C, X')
\end{tikzcd}$$

$\mapsto$

$$\begin{tikzcd}
F(C) \arrow[r, "\mathrm{inc}_C"] \arrow[d, "F(f)"'] \arrow[d, phantom, shift left=6ex, "\Downarrow \mathrm{inc}_f"] & \int F \\
F(D) \arrow[ru, bend right, "\mathrm{inc}_D"'] &
\end{tikzcd}$$

$$(\mathrm{inc}_f)_X = (f, \mathrm{id})$$

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& X' & & (C, X')
\end{tikzcd}$$

$$\begin{tikzcd}
F(C) \arrow[rr, "\mathrm{inc}_C"] \arrow[d, "F(f)"'] & \arrow[d, Rightarrow, "\mathrm{inc}_f"] & \int F \\
F(D) \arrow[urr, bend right, "\mathrm{inc}_D"'] & &
\end{tikzcd}$$

$$(\mathrm{inc}_f)_X = (f, \mathrm{id})$$

$$\begin{tikzcd}
\mathcal{E} \arrow[rr] \arrow[d, "\forall p \text{ opfibration}"'] & & \mathbb{1} \arrow[d, "\mathbb{1}"] \arrow[dll, Rightarrow, "\text{lax comma}"] \\
\mathcal{C} \arrow[rr, dashed, "\exists \chi_p"', "\text{taking fibres}"'] & & \mathcal{C}at
\end{tikzcd}$$

equivalence of categories

$$[\mathcal{C}, \mathcal{CAT}] \simeq \mathcal{OpFib}(\mathcal{C})$$

$$[\mathcal{C}, \mathcal{S}et] \simeq \mathcal{DOpFib}(\mathcal{C})$$

Opfibrations in a 2-category $\mathcal{L}$

Definition.

A **split opfibration in $\mathcal{L}$** is $\varphi\colon G \to F$ in $\mathcal{L}$ s.t. for every $X \in \mathcal{L}$

$$\varphi \circ -\colon \mathcal{L}(X, G) \to \mathcal{L}(X, F)$$

is a split opfibration (in $\mathcal{CAT}$) and for every $\lambda\colon K \to X$ in $\mathcal{L}$

$$\begin{tikzcd} \mathcal{L}(X,G) \arrow[r, "-\circ\lambda"] \arrow[d, "\varphi\circ-"'] & \mathcal{L}(K,G) \arrow[d, "\varphi\circ-"] \\ \mathcal{L}(X,F) \arrow[r, "-\circ\lambda"'] & \mathcal{L}(K,F) \end{tikzcd}$$

is cleavage preserving.

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\mathcal{L}(X,F) \arrow[r, "-\circ\lambda"'] & \mathcal{L}(K,F)
\end{tikzcd}$$

is cleavage preserving.

In $\mathcal{CAT}$, it suffices to look at $X = 1$

$$\begin{tikzcd}
X \arrow[rr, bend left, "\alpha"] \arrow[rr, bend right, "\theta_*\alpha"'] \arrow[rr, phantom, "\Downarrow \bar{\theta}^{\alpha}"] \arrow[d, equal] & & G \arrow[d, "\varphi"] \\
X \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta"'] \arrow[rr, phantom, "\Downarrow \theta"'] & G \arrow[r, "\varphi"] & F
\end{tikzcd}$$

Opfibrations in $[\mathcal{A}, \mathcal{CAT}]$

Proposition (Caviglia–M.).

Let $\mathcal{A}$ be a category. The following facts are equivalent:

(i) $\varphi\colon G \to F$ is a **split opfibration in $[\mathcal{A}, \mathcal{CAT}]$**;

(ii) **every component $\varphi_A\colon G(A) \to F(A)$ is a split opfibration** (in $\mathcal{CAT}$) and for every $h\colon A \to B$ in $\mathcal{A}$ the naturality square

$$\begin{tikzcd} G(A) \arrow[r, "G(h)"] \arrow[d, "\varphi_A"'] & G(B) \arrow[d, "\varphi_B"] \\ F(A) \arrow[r, "F(h)"'] & F(B) \end{tikzcd}$$

is cleavage preserving.

Idea: it suffices to look at the representables.

Indexed Grothendieck construction

Theorem (Caviglia–M.).

For every 2-functor $F\colon \mathcal{A} \to \mathcal{CAT}$, there is an **equivalence of categories**

$$\mathcal{OpFib}_{[\mathcal{A},\mathcal{CAT}]}(F) \simeq \left[\int F, \mathcal{CAT}\right]$$

This restricts to an equivalence of categories

$$\mathcal{DOpFib}_{[\mathcal{A},\mathcal{CAT}]}(F) \simeq \left[\int F, \mathcal{Set}\right]$$

Moreover, both the equivalences are pseudonatural in F.

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Moreover, both the equivalences are pseudonatural in F.

When $\mathcal{A} = 1$, we recover the usual Grothendieck construction.

Indeed $F\colon 1 \to \mathcal{CAT}$ is just a category $\mathcal{C}$ and $\int F = \mathcal{C}$.

$$\mathcal{OpFib}(\mathcal{C}) \simeq [\mathcal{C}, \mathcal{CAT}] .$$

Indexed Grothendieck construction

$$\mathcal{DOpFib}_{[\mathcal{A},\mathcal{CAT}]}(F) \simeq \left[\int F, \mathcal{S}et\right]$$

further reduces, **when $F\colon \mathcal{A} \to \mathcal{S}et$**, to the well-known

$$[\mathcal{A}, \mathcal{S}et]/_F \simeq \left[\int F, \mathcal{S}et\right].$$

When F is a representable $\mathrm{y}(A) : \mathcal{A} \to \mathcal{S}et$, this is the famous

$$[\mathcal{A}, \mathcal{S}et]\Big/_{\mathrm{y}(A)} \simeq [\mathcal{A}/_A, \mathcal{S}et].$$

We find the Grothendieck topoi case of the **fundamental theorem of elementary topos theory**.

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$$[\mathcal{A}, \mathcal{Set}]\Big/_{\mathrm{y}(A)} \simeq [\mathcal{A}/_A, \mathcal{Set}].$$

We find the Grothendieck topoi case of the **fundamental theorem of elementary topos theory**. Our equivalence

$$\mathcal{OpFib}_{[\mathcal{A},\mathcal{CAT}]}(F) \simeq \left[\int F, \mathcal{CAT}\right]$$

shows that **every (op)fibrational slice of a Grothendieck 2-topos is again a Grothendieck 2-topos**.

Indexed Grothendieck construction

Idea of the proof:

$$\left[\int F, \mathcal{C}at\right] \cong [\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}\left(\Delta 1, [F(-), \mathcal{C}at]\right) \simeq$$

$$\simeq \mathrm{Ps}\,[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}\left(\Delta 1, \mathcal{O}p\mathcal{F}ib_{\mathcal{CAT}}(F(-))\right) \cong \mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{CAT}]}(F)$$

$$\int F = \mathrm{oplax}\text{-}\mathrm{colim}\, F$$

$$\mathcal{G}_{-} \colon [-, \mathcal{C}at] \simeq \mathcal{O}p\mathcal{F}ib(-)$$

$$\begin{tikzcd}
G(A) \arrow[rrd, bend left, "G(h)"] \arrow[rd, dashed, "{[\varphi]_h}"] \arrow[rdd, bend right, "\varphi_A"'] & & \\
& F(h)^*(G(B)) \arrow[r] \arrow[d, "F(h)^*(\varphi_B)"'] \arrow[rd, phantom, "\lrcorner", very near start] & G(B) \arrow[d, "\varphi_B"] \\
& F(A) \arrow[r, "F(h)"'] & F(B)
\end{tikzcd}$$

Explicit indexed Grothendieck construction

An opfibration $\varphi\colon G \to F$ in $[\mathcal{A}, \mathcal{CAT}]$ is sent to

$$\begin{tikzcd}
\int F \arrow[rr] & & \mathcal{CAT} \\
(A,X) \arrow[d, "{(h,\mathrm{id})}"] \arrow[dd, bend right=60, "{(h,\alpha)}"'] & & (\varphi_A)_X \arrow[d, "G(h)"] \\
(B, F(f)(X)) \arrow[d, "{(\mathrm{id},\alpha)}"] & \mapsto & (\varphi_B)_{F(h)(X)} \arrow[d, "\alpha_*"] \\
(B, X') & & (\varphi_B)_{X'}
\end{tikzcd}$$

Explicit indexed Grothendieck construction

An opfibration $\varphi\colon G \to F$ in $[\mathcal{A}, \mathcal{CAT}]$ is sent to

$$\begin{tikzcd}
\int F \arrow[rr, "\longrightarrow", phantom] & & \mathcal{CAT} \\
(A,X) \arrow[d, "{(h,\mathrm{id})}"] \arrow[dd, bend right=60, "{(h,\alpha)}"'] & & (\varphi_A)_X \arrow[d, "G(h)"] \\
(B,F(f)(X)) \arrow[d, "{(\mathrm{id},\alpha)}"] & \mapsto & (\varphi_B)_{F(h)(X)} \arrow[d, "\alpha_*"] \\
(B,X') & & (\varphi_B)_{X'}
\end{tikzcd}$$

A 2-functor $Z\colon \int F \to \mathcal{CAT}$ is sent to the opfibration whose component $G(A) \to F(A)$ is the projection on the first component with

an object of $G(A)$ is a **pair (X,ξ)** with $X \in F(A)$ and $\xi \in Z(A,X)$;

a morphism $(X,\xi) \to (X',\xi')$ *in* $G(A)$ is a **pair (α,Λ)** with $\alpha\colon X \to X'$ in $F(A)$ and $\Lambda\colon Z(\mathrm{id},\alpha)(\xi) \to \xi'$ in $Z(A,X')$.

Example ($\mathcal{A} = 2$).

When $\mathcal{A} = 2$, we have $[\mathcal{A}, \mathcal{CAT}] = \mathcal{CAT}^2$, so that $F\colon 2 \to \mathcal{CAT}$ is a functor $\widetilde{F}\colon \mathcal{C} \to \mathcal{D}$. A split opfibration $\varphi\colon G \to F$ in $[2, \mathcal{CAT}]$ is **a cleavage preserving morphism between split opfibrations in $\mathcal{CAT}$**

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\widetilde{G}"] \arrow[d, "\varphi_0"'] & \mathcal{L} \arrow[d, "\varphi_1"] \\
\mathcal{C} \arrow[r, "\widetilde{F}"'] & \mathcal{D}
\end{tikzcd}$$

This can be **reorganized as a 2-functor $\int F \to \mathcal{CAT}$**. The objects of $\int F$ are the disjoint union of the objects of $\mathcal{C}$ and of $\mathcal{D}$. The morphisms in $\int F$ are of three kinds: morphisms in $\mathcal{C}$ (over 0), morphisms in $\mathcal{D}$ (over 1) and morphisms over $0 \to 1$ that represents the objects $(C, D, \widetilde{F}(C) \to D)$ of the comma category $\widetilde{F}/_{\mathcal{D}}$.

Example ($\mathcal{A} = \Delta$).

When $\mathcal{A}$ is the simplex category Δ, we have that $F\colon \Delta \to \mathcal{CAT}$ is a cosimplicial category. **Split opfibrations between cosimplicial categories over F** are equivalent to **2-copresheaves on the total category that collects all the cosimplexes given by F**.

Example (semidirect product of groups).

Let $\mathcal{A}$ be the one-object category $\mathcal{B}G$ corresponding to a group G. A functor $F\colon \mathcal{B}G \to \mathcal{CAT}$ which picks some $\mathcal{B}H$ corresponds with $\rho\colon G \to \mathrm{Aut}(H)$. Then $\int F = \mathcal{B}(H \rtimes_\rho G)$. **Functors $\mathcal{B}(H \rtimes_\rho G) \to \mathcal{CAT}$** are equivalent to **opfibrations in $[\mathcal{B}G, \mathcal{CAT}]$ over the functor F that corresponds with $\rho\colon G \to \mathrm{Aut}(H)$**.

Example.

2-classifiers, introduced by Weber, are the generalization to dimension 2 of subobject classifiers. **$\mathcal{CAT}$ is the archetypal elementary 2-topos**. Its 2-dimensional classification process is the Grothendieck construction.

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The indexed Grothendieck construction is a key element to show a 2-classifier in 2-presheaves.

$$\Omega(A) = \mathcal{DOpFib}_{[\mathcal{A}^{\mathrm{op}},\mathcal{CAT}]}(\mathrm{y}(A)) \simeq \left[\left(\mathcal{A}/_{A}\right)^{\mathrm{op}}, \mathcal{Set}\right] = \widetilde{\Omega}(A)$$

$\widetilde{\Omega}$ is a strictly functorial replacement of Ω.

Theorem (M.).

$\widetilde{\Omega}$ is a 2-classifier in $[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]$ that classifies all discrete opfibrations with small fibres. It can be also **restricted to a 2-classifier in stacks**.