Colimits in 2-dimensional slices

Luca Mesiti

University of Leeds

PSSL 107 in Athens

02/04/2023

Colimits in 1-dimensional slices

Theorem.

Let $\mathcal{C}$ be a category and let $M \in \mathcal{C}$.

$\mathrm{dom}\colon \mathcal{C}/_M \to \mathcal{C}$ preserves, reflects and lifts uniquely all the colimits.

$$\begin{array}{c} \operatorname{colim}_A D(A) \\ \downarrow q \\ M \end{array} \cong \operatorname{colim}_A \begin{array}{c} D(A) \\ \downarrow q \circ i_A \\ M \end{array} \quad \text{in } \mathcal{C}/_M,$$

where the i_A's are the inclusions of the components $D(A)$ in their colimit.

Colimits in 1-dimensional slices

Theorem.

Let $\mathcal{C}$ be a category and let $M \in \mathcal{C}$.

dom: $\mathcal{C}/_M \to \mathcal{C}$ preserves, reflects and lifts uniquely all the colimits.

$$\begin{array}{c} \operatorname{colim}_A D(A) \\ \downarrow q \\ M \end{array} \cong \operatorname{colim}_A \begin{array}{c} D(A) \\ \downarrow q \circ i_A \\ M \end{array} \quad \text{in } \mathcal{C}/_M,$$

where the i_A's are the inclusions of the components $D(A)$ in their colimit.

This is based on the fact that a cocone on M is the same thing as a diagram in $\mathcal{C}/_M$.

Colimits in 2-dimensional slices

Let $\mathcal{E}$ be a 2-category and let $M \in \mathcal{E}$.

Consider $F\colon \mathcal{A} \to \mathcal{E}$ and $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{CAT}$ such that $\mathrm{colim}^W F$ exists.

Take then $q\colon \mathrm{colim}^W F \to M$, or equivalently

$$\nu^q\colon W \Rightarrow \mathcal{E}\left(F(-), M\right).$$

We want to express q as a 2-colimit in a 2-dimensional slice of $\mathcal{E}$ on M.

Colimits in 2-dimensional slices

Let $\mathcal{E}$ be a 2-category and let $M \in \mathcal{E}$.
Consider $F\colon \mathcal{A} \to \mathcal{E}$ and $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{CAT}$ such that $\operatorname{colim}^W F$ exists.

Take then $q\colon \operatorname{colim}^W F \to M$, or equivalently

$$\nu^q\colon W \Rightarrow \mathcal{E}\left(F(-), M\right).$$

We want to express q as a 2-colimit in a 2-dimensional slice of $\mathcal{E}$ on M.

This is essentially a matter of selecting a cocone out of the bunch of cocones that form the weighted 2-cocylinder ν^q.
We reduce weighted 2-colimits to oplax normal conical ones!

$$\operatorname{colim}^W F \cong \text{oplax}^{\mathrm{n}}\text{-}\operatorname{colim}^{\Delta 1}(F \circ \mathcal{G}(W))$$

$\mathcal{G}(W)\colon \int W \to \mathcal{A}$ is the 2-$\mathcal{S}et$-enriched Grothendieck construction of W.

And ν^q corresponds to an oplax normal 2-cocone

$$\lambda^q \colon \Delta 1 \underset{\text{oplax}^\text{n}}{\Longrightarrow} \mathcal{E}((F \circ \mathcal{G}(W))(-), M) : \left(\int W\right)^{\text{op}} \to \mathcal{CAT}.$$

"Normal": for every (f, id) in $\left(\int W\right)^{\text{op}}$ the structure 2-cell $\lambda^q_{(f,\text{id})} = \text{id}$.

We can reorganize λ^q as the 2-diagram

$$\begin{array}{cccc} L^q : & \int W & \longrightarrow & \mathcal{E}/_{\text{lax}}\, M \\ & \begin{array}{c}(A, X) \\ \downarrow{\scriptstyle (f,\alpha)} \\ (B, X')\end{array} & \mapsto & \begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ & \overset{\lambda^q_{f,\alpha}}{\Longrightarrow} & \\ {\scriptstyle \lambda^q_{(A,X)}} \searrow & & \swarrow {\scriptstyle \lambda^q_{(B,X')}} \\ & M & \end{array} \end{array}$$

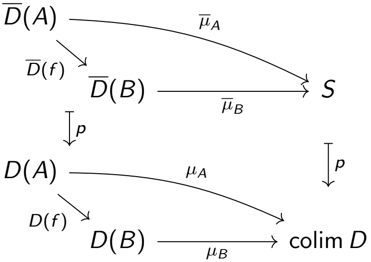
We will see later that

$$\begin{array}{c}\text{colim}^W F \\ \downarrow q \\ M\end{array} = \begin{array}{c}\text{oplax}^\text{n}\text{-colim}^{\Delta 1}(F \circ \mathcal{G}(W)) \\ \downarrow q \\ M\end{array} = \text{oplax}^\text{n}\text{-colim}^{\Delta 1} L^q$$

Colim-fibrations

Definition (M.).

A functor $p\colon \mathcal{S} \to \mathcal{C}$ is a **colim-fibration** if for every $S \in \mathcal{S}$ and every universal cocone μ that exhibits $p(S) = \operatorname{colim} D$, there exists a unique pair $(\overline{D}, \overline{\mu})$ with $\overline{D}\colon \mathcal{A} \to \mathcal{S}$ a diagram and $\overline{\mu}$ a universal cocone that exhibits $S = \operatorname{colim} \overline{D}$ such that $p \circ \overline{D} = D$ and $p \circ \overline{\mu} = \mu$.

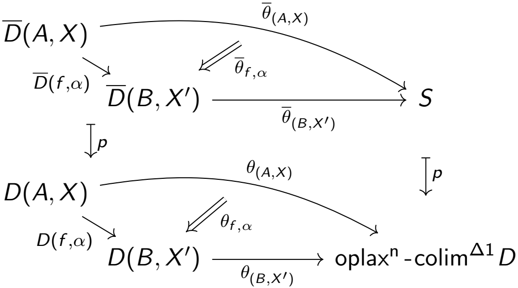


Equivalently, p is a discrete fibration that reflects all the colimits.

2-colim-fibrations

Discrete fibrations $\rightsquigarrow$ 2-$\mathcal{S}et$-fibrations.

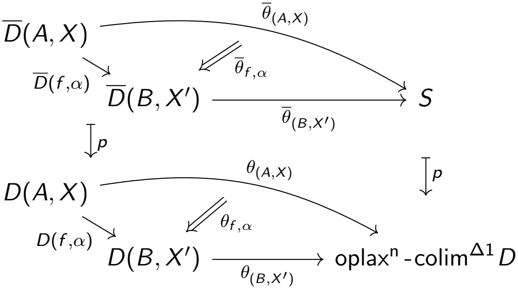
They are fibrations that also uniquely lift 2-cells to a fixed domain 1-cell.



2-colim-fibrations

Discrete fibrations $\rightsquigarrow$ 2-$\mathcal{S}et$-fibrations.

They are fibrations that also uniquely lift 2-cells to a fixed domain 1-cell.



Definition (M.).

$p\colon \mathcal{S} \to \mathcal{E}$ is a 2-**colim-fibration** if it is a cloven 2-$\mathcal{S}et$-fibration such that, for every S, D, θ, the pair $(\overline{D}, \overline{\theta})$ exhibits $S = \mathrm{oplax}^{\mathrm{n}}\text{-}\mathrm{colim}^{\Delta 1}\overline{D}$.

Equiv., a cloven 2-$\mathcal{S}et$-fibration that reflects all the cartesian 2-colimits.

Colimits in 2-dimensional slices, via 2-colim-fibrations

$\mathcal{S}et$-enriched

$$
\begin{array}{ccc}
\mathcal{C}/M & \longrightarrow & \mathit{1} \\
{\scriptstyle \mathrm{dom}}\downarrow & \overset{\text{comma}}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle \mathit{1}} \\
\mathcal{E} & \xrightarrow[\mathrm{y}(M)]{} & \mathcal{S}et^{\mathrm{op}}
\end{array}
$$

2-$\mathcal{S}et$-enriched

$$
\begin{array}{ccc}
\mathcal{E}/_{\mathrm{lax}}\, M & \longrightarrow & \mathit{1} \\
{\scriptstyle \mathrm{dom}}\downarrow & \overset{\text{lax comma}}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle \mathit{1}} \\
\mathcal{E} & \xrightarrow[\mathrm{y}(M)]{} & \mathcal{CAT}^{\mathrm{op}}
\end{array}
$$

Colimits in 2-dimensional slices, via 2-colim-fibrations

$\mathcal{S}et$-enriched

$$\begin{array}{ccc} \mathcal{C}/M & \longrightarrow & \mathit{1} \\ {\scriptstyle \mathrm{dom}}\Big\downarrow & \overset{\text{comma}}{\Nearrow} & \Big\downarrow{\scriptstyle \mathit{1}} \\ \mathcal{E} & \underset{\mathrm{y}(M)}{\longrightarrow} & \mathcal{S}et^{\mathrm{op}} \end{array}$$

2-$\mathcal{S}et$-enriched

$$\begin{array}{ccc} \mathcal{E}/_{\mathrm{lax}}\, M & \longrightarrow & \mathit{1} \\ {\scriptstyle \mathrm{dom}}\Big\downarrow & \overset{\text{lax comma}}{\Nearrow} & \Big\downarrow{\scriptstyle \mathit{1}} \\ \mathcal{E} & \underset{\mathrm{y}(M)}{\longrightarrow} & \mathcal{CAT}^{\mathrm{op}} \end{array}$$

Theorem (M.).

$\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ is a 2-colim-fibration. As a consequence,

$$\begin{array}{c} \mathrm{colim}^{W} F \\ \downarrow q \\ M \end{array} = \begin{array}{c} \mathrm{oplax}^{\mathrm{n}}\text{-}\mathrm{colim}^{\Delta 1}(F \circ \mathcal{G}(W)) \\ \downarrow q \\ M \end{array} = \mathrm{oplax}^{\mathrm{n}}\text{-}\mathrm{colim}^{\Delta 1} L^{q}$$

$$\overline{F \circ \mathcal{G}(W)}(A, X) = \mathrm{y}(M)\,(\theta_{(A,X)})(q) = q \circ \theta_{(A,X)} = \lambda^{q}_{(A,X)} = L^{q}(A, X)$$

dom lifts 2-colimits of $\mathcal{F}$-diagrams

Consider a marking $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{CAT}$ and a 2-diagram $D\colon \int W \to \mathcal{E}$. There is a bijection between oplax normal 2-cocones

$$\lambda\colon \Delta 1 \underset{\text{oplax}^{\text{n}}}{\Longrightarrow} \mathcal{E}\,(D(-), M)$$

on M and 2-diagrams $\overline{D}\colon \int W \to \mathcal{E}\,/_{\text{lax}}\, M$ such that for every morphism (f, id) in $\int W$ the triangle $\overline{D}(f, \mathrm{id})$ is filled with an identity.

dom lifts 2-colimits of $\mathcal{F}$-diagrams

Consider a marking $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{CAT}$ and a 2-diagram $D\colon \int W \to \mathcal{E}$. There is a bijection between oplax normal 2-cocones

$$\lambda\colon \Delta 1 \xRightarrow[\mathrm{oplax^n}]{} \mathcal{E}(D(-), M)$$

on M and 2-diagrams $\overline{D}\colon \int W \to \mathcal{E}/_{\mathrm{lax}}\, M$ such that for every morphism (f, id) in $\int W$ the triangle $\overline{D}(f, \mathrm{id})$ is filled with an identity.

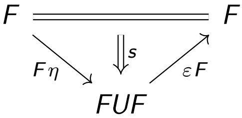
That is, $\mathcal{F}$-**diagrams** $\overline{D}\colon \int W \to \mathcal{E}/_{\mathrm{lax}}\, M$, taking on $\mathcal{E}/_{\mathrm{lax}}\, M = \int \mathrm{y}(M)$ the canonical $\mathcal{F}$-category structure, with loose part $\mathcal{E}/_{\mathrm{lax}}\, M$ itself and tight part the strict 2-slice $\mathcal{E}/_{M}$.

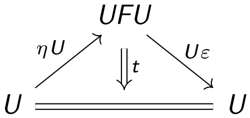
dom: $\mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ lifts all the oplax normal 2-colimits of $\mathcal{F}$-diagrams, since an $\mathcal{F}$-diagram corresponds to an oplax normal 2-cocone, that induces a unique morphism $q\colon C \to M$.

Towards preservation of 2-colimits: lax adjunctions

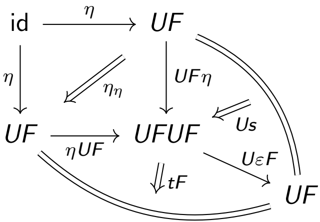
Definition (Gray).

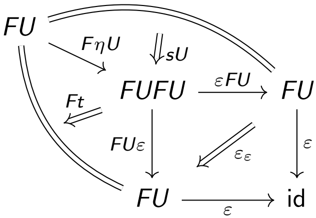
A **lax adjunction** from $F\colon \mathcal{A} \to \mathcal{B}$ to $U\colon \mathcal{B} \to \mathcal{A}$ is given by a lax natural unit $\eta\colon \mathrm{Id} \Rightarrow UF$, a lax natural counit $\varepsilon\colon FU \Rightarrow \mathrm{Id}$ and modifications





(lax triangular laws) such that both the swallowtails are identities:



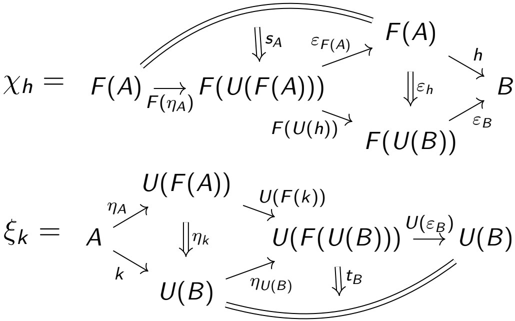


Right semi-lax adjunction when ε is strictly 2-natural and $s = \mathrm{id}$.

Strict when s and t are both identities, so triangular laws hold strictly.

A lax adjunction gives in particular oridinary adjunctions between homsets

$$\mathcal{B}(F(A), B) \underset{T}{\overset{S}{\rightleftarrows}} \mathcal{A}(A, U(B)) \qquad \perp_{\chi,\xi}$$



And if the lax adjunction is right semi-lax then $T \circ S = \mathrm{id}$.

Definition (Walker, M.).

A **loose lax $\mathcal{F}$-adjunction** is a lax adjunction $(F, U, \eta, \varepsilon, s, t)$ between the loose parts in which F and U are $\mathcal{F}$-functors and η and ε are loose strict/lax $\mathcal{F}$-natural transformations.

(**Tight**) **lax $\mathcal{F}$-adjunction** when η and ε have tight components.

Definition (M.).

Consider $\mathcal{F}$-functors $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{F}$ and $H\colon \mathcal{A} \to \mathcal{S}$.

The **strict/oplax $\mathcal{F}$-colimit** of H weighted by W is $C \in \mathcal{S}$ with 2-natural

$$\mathcal{S}_\lambda\left(C, Q\right) \cong \left[\mathcal{A}_\lambda^{\mathrm{op}}, \mathcal{CAT}\right]_{\mathrm{oplax}^{\mathrm{n}}}\left(W_\lambda, \mathcal{S}_\lambda\left(H_\lambda(-), Q\right)\right),$$

s.t. the "components" $\mu_A^\lambda(X)$'s of the universal oplax normal cocylinder, for $A \in \mathcal{A}$ and $X \in W_\tau(A)$, are tight and jointly detect tightness.

Right semi-lax left $\mathcal{F}$-adjoints preserve colimits

Theorem (M.).

Right semi-lax loose left $\mathcal{F}$-adjoints preserve all the universal oplax normal 2-cocylinders for an $\mathcal{F}$-diagram which have tight "components", even if such "components" do not jointly detect tightness.

Right semi-lax (tight) left $\mathcal{F}$-adjoints preserve all the strict/oplax $\mathcal{F}$-colimits.

$$
\begin{array}{ccccc}
W & \overset{\mu}{\Longrightarrow} & \mathcal{S}\,(H(-),C) & \overset{F}{\Longrightarrow} & \mathcal{E}\,((F\circ H)(-),F(C)) \\
\Big\Downarrow{\scriptstyle\sigma} & & \Big\Downarrow{\scriptstyle\gamma\circ -} & & \Big\Downarrow{\scriptstyle T(\gamma)\circ -} \\
\mathcal{E}\,((F\circ H)(-),Z) & \underset{S}{\Longrightarrow} & \mathcal{S}\,(H(-),U(Z)) & \underset{T}{\Longrightarrow} & \mathcal{E}\,((F\circ H)(-),Z)
\end{array}
$$

dom preserves all the strict/oplax $\mathcal{F}$-colimits

Theorem (M.).

dom: $\mathcal{E}/_{\text{lax}}\, M \to \mathcal{E}$ has a strict right semi-lax tight right $\mathcal{F}$-adjoint U.

As a consequence, dom preserves all the strict/oplax $\mathcal{F}$-colimits, but also all the universal oplax normal 2-cocones for an $\mathcal{F}$-diagram which have tight components.

For every $E \in \mathcal{E}$, we define

$$U(E) \coloneqq (M \times E \xrightarrow{\text{pr}_1} E)$$

$$\varepsilon_E \colon M \times E \xrightarrow{\text{pr}_2} E$$

The counit ε_E is universal in a lax sense, reminiscent of Kan extensions.

Change of base between lax slices

Theorem (M.).

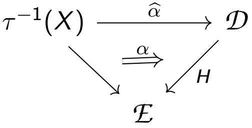
Let $\tau\colon \mathcal{E} \to \mathcal{B}$ be a split Grothendieck opfibration. The pullback 2-functor

$$\tau^*\colon \mathcal{CAT}/_{\mathrm{lax}}\,\mathcal{B} \to \mathcal{CAT}/_{\mathrm{lax}}\,\mathcal{E}$$

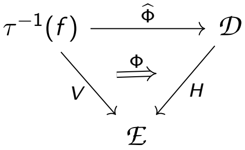
has a strict right semi-lax loose right $\mathcal{F}$-adjoint τ_*.

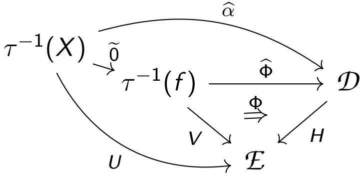
As a consequence, τ^* preserves all the universal oplax normal 2-cocylinders for an $\mathcal{F}$-diagram which have tight "components".

For every $H\colon \mathcal{D} \to \mathcal{E}$, we define $\tau_* H$ as $\mathrm{pr}_1\colon \mathcal{H} \to \mathcal{B}$, where the category $\mathcal{H}$ has as objects pairs $(X, (\widehat{\alpha}, \alpha))$ with $X \in \mathcal{B}$ and

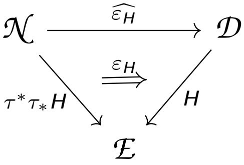


A morphism $(X,(\widehat{\alpha},\alpha)) \to (X',(\widehat{\beta},\beta))$ in $\mathcal{H}$ is a pair $(f,(\widehat{\Phi},\Phi))$ with $f\colon X \to X'$ in $\mathcal{B}$ and $(\widehat{\Phi},\Phi)$ as below such that $\Phi * \widetilde{0} = \alpha$ and $\Phi * \widetilde{1} = \beta$





The counit is an evaluation: on every $H\colon \mathcal{D} \to \mathcal{E}$



$$\widehat{\varepsilon_H}((X,(\widehat{\alpha},\alpha)),E) \coloneqq \widehat{\alpha}(E)$$

$$\widehat{\varepsilon_H}((f,(\widehat{\Phi},\Phi)),g) \coloneqq \widehat{\Phi}(0 \to 1, g)$$

$$(\varepsilon_H)_{((X,(\widehat{\alpha},\alpha)),E)} \coloneqq \alpha_E$$