

Lax normal conical 2-limits and the Grothendieck construction

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Itaca Workshop

21/12/2022

Weighted enriched limits

$F: \mathcal{A} \rightarrow \mathcal{C}$ with \mathcal{A} small

An ordinary cone is a natural transformation

$$\Delta U \Rightarrow F: A \rightarrow \mathcal{C}$$

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But 1 will not be sensitive enough now.

Consider a weight $W: \mathcal{A} \rightarrow \mathcal{V}$

$$C\left(U, \lim^W F\right) \cong [\mathcal{A}, \mathcal{V}](W, C(U, F(-)))$$

We call it **conical** when $W = \Delta 1$.

Conicalization of weighted \mathcal{V} -limits

Example 1 (Every presheaf is a colimit of representables).

$W: \mathcal{A} \rightarrow \mathcal{V}$ with \mathcal{A} small. Consider $y: \mathcal{A}^{\text{op}} \rightarrow [\mathcal{A}, \mathcal{V}]$

$$W \cong \text{colim}^W y$$

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Example 1 (Every presheaf is a colimit of representables).

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Lemma 2 (Continuity in the weight).

$S: \mathcal{B}^{\text{op}} \rightarrow \mathcal{V}$, $Z: \mathcal{B} \rightarrow [\mathcal{A}, \mathcal{V}]$, $F: \mathcal{A} \rightarrow \mathcal{C}$

Suppose that $\text{colim}^S Z$ and each $\lim^{Z(B)} F$ with $B \in \mathcal{B}$ exist. Then

$$\lim^{\text{colim}^S Z} F \cong \lim^S \left(\lim^{Z(-)} F \right)$$

Conicalization of weighted 2-limits

$W: \mathcal{A} \rightarrow \mathcal{CAT}$ with \mathcal{A} small. We search for $H: \mathcal{B} \rightarrow [\mathcal{A}, \mathcal{CAT}]$

$$\frac{\varphi: W \Rightarrow [\mathcal{A}, \mathcal{CAT}](y(-), U)}{\tilde{\varphi}: \Delta 1 \xRightarrow[\text{relaxed}]{} [\mathcal{A}, \mathcal{CAT}](H(-), U) : \mathcal{B}^{\text{op}} \rightarrow \mathcal{CAT}}$$

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As we will want to apply the lemma of continuity in the weight, we search for H of the form

$$\left(\int^{\text{op}} W\right)^{\text{op}} \xrightarrow{\mathcal{G}(W)^{\text{op}}} \mathcal{A}^{\text{op}} \xrightarrow{y} [\mathcal{A}, \mathcal{CAT}].$$

And we want $\tilde{\varphi}$ to be at least a lax natural transformation.

Conicalization of weighted 2-limits

$$\frac{\varphi: W \Rightarrow [\mathcal{A}, \mathcal{CAT}](y(-), U)}{\tilde{\varphi}: \Delta 1 \xrightarrow[\text{relaxed}]{} [\mathcal{A}, \mathcal{CAT}]((y \circ \mathcal{G}(W)^{\text{op}})(-), U) : \int^{\text{op}} W \rightarrow \mathcal{CAT}}$$

For every $A \in \mathcal{A}$ and $X \in W(A)$, we have a morphism

$$\varphi_A(X): y(A) \rightarrow U,$$

so we need an object $(A, X) \in \int^{\text{op}} W$ such that $\mathcal{G}(W)(A, X) = A$.

Conicalization of weighted 2-limits

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For every $\alpha: X \rightarrow X'$ in $W(A)$, we encode the 2-cell $\varphi_A(\alpha)$ as

$$\begin{array}{ccc} (A, X) & 1 & \xrightarrow{\tilde{\varphi}_{(A, X)}} [\mathcal{A}, \mathcal{CAT}](y(A), U) \\ \downarrow (\text{id}_A, \alpha) & \parallel & \tilde{\varphi}_{(\text{id}_A, \alpha)} \swarrow \quad \parallel_{-\circ y(\mathcal{G}(W)^{\text{op}}(\text{id}_A, \alpha))} \\ (A, X') & 1 & \xrightarrow{\tilde{\varphi}_{(A, X')}} [\mathcal{A}, \mathcal{CAT}](y(A), U) \end{array}$$

Conicalization of weighted 2-limits

In order to encode the naturality of φ into the relaxed naturality of $\tilde{\varphi}$,
for every $f: A \rightarrow A'$ in \mathcal{A} and $X \in W(A)$,

$$\begin{array}{ccc}
 (A, X) & \xrightarrow{\tilde{\varphi}_{(A, X)}} & [\mathcal{A}, \mathcal{CAT}](y(A), U) \\
 \downarrow \underline{f^X} = (f, \text{id}) & \parallel \quad \tilde{\varphi}_{\underline{f^X}} \quad \quad \quad \text{--oy}(f) \downarrow \text{--oy}(\mathcal{G}(W)^{\text{op}}(\underline{f^X})) & \\
 (A', W(f)(X)) & \xrightarrow{\tilde{\varphi}_{(A', W(f)(X))}} & [\mathcal{A}, \mathcal{CAT}](y(A'), U)
 \end{array}$$

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 \downarrow \underline{f}^X = (f, \text{id}) & \parallel & \tilde{\varphi}_{\underline{f}^X} \cong \text{--oy}(f) \downarrow \text{--oy}(\mathcal{G}(W)^{\text{op}}(\underline{f}^X)) \\
 (A', W(f)(X)) & \xrightarrow{\tilde{\varphi}_{(A', W(f)(X))}} & [\mathcal{A}, \mathcal{CAT}](y(A'), U)
 \end{array}$$

$$\underline{f}^{X'} \circ (\text{id}_A, \alpha) = (\text{id}_{A'}, W(f)(\alpha)) \circ \underline{f}^X$$

so that we recover the naturality of φ on morphisms, that is

$$\varphi_A(\alpha) \circ y(f) = \varphi_{A'}(W(f)(\alpha)).$$

At this point, every finite composition is reduced to (f, α) defined as

$$(A, X) \xrightarrow{\underline{f}^X} (A', W(f)(X)) \xrightarrow{(\text{id}_{A'}, \alpha)} (A', X')$$

Conicalization of weighted 2-limits

In order to encode the 2-naturality of φ , that is
for every $\delta: f \Rightarrow g: A \rightarrow A'$ in \mathcal{A} and every $X \in W(A)$

$$\varphi_{A'}(W(\delta)_X) = \varphi_A(X) y(\delta),$$

we need a 2-cell $\underline{\delta}^X$ with $\mathcal{G}(W)(\underline{\delta}^X) = \delta$

$$\underline{\delta}^X: (f, W(\delta)_X) \Rightarrow \underline{g}^X: (A, X) \rightarrow (A', W(g)(X))$$

$$\begin{array}{ccc} 1 \xrightarrow{\tilde{\varphi}_{(A,X)}} [\mathcal{A}, \mathcal{CAT}](y(A), U) & & 1 \xrightarrow{\tilde{\varphi}_{(A,X)}} [\mathcal{A}, \mathcal{CAT}](y(A), U) \\ \parallel & \swarrow \tilde{\varphi}_{(f, W(\delta)_X)} \downarrow -\text{oy}(f) & \parallel & \swarrow \tilde{\varphi}_{\underline{g}^X} \downarrow -\text{oy}(g) \\ 1 \xrightarrow{\tilde{\varphi}_{(A', W(g)(X))}} [\mathcal{A}, \mathcal{CAT}](y(A'), U) & = & 1 \xrightarrow{\tilde{\varphi}_{(A', W(g)(X))}} [\mathcal{A}, \mathcal{CAT}](y(A'), U) \end{array}$$

$\begin{array}{c} \text{---} * y(\mathcal{G}(W)^{\text{op}}(\underline{\delta}^X)) \text{---} \\ \text{---} \text{oy}(g) \text{---} \end{array} \begin{array}{c} \text{---} \text{oy}(f) \text{---} \\ \text{---} \text{oy}(f) \text{---} \end{array}$

Conicalization of weighted 2-limits

$$\underline{\delta}^{X'}(\mathrm{id}_A, \alpha) = (\mathrm{id}_{A'}, W(g)(\alpha))\underline{\delta}^X$$

because they represent the same axiom. At this point, every horizontal composition of 2-cells is reduced to a whiskering

$$\delta := (\mathrm{id}, \beta)\underline{\delta}^X : (f, \beta \circ W(\delta)_X) \Rightarrow (g, \beta).$$

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We call $\mathcal{G}(W)$ (or sometimes also just $\int^{\mathrm{op}} W$) the **2-Set-enriched Grothendieck construction** of $W: \mathcal{A} \rightarrow \mathcal{CAT}$.

A **lax normal natural transformation** $\alpha: M \xRightarrow[\mathrm{lax}^n]{} N: \int^{\mathrm{op}} W \rightarrow \mathcal{D}$ is a lax natural transformation such that its structure 2-cell on every

$$\underline{f}^X: (A, X) \rightarrow (B, W(f)(X))$$

in $\int^{\mathrm{op}} W$ is the identity.

Conicalization of weighted 2-limits

Definition 3 ((Op)lax normal conical 2-(co)limit).

$W: \mathcal{A} \rightarrow \mathcal{CAT}$ and $F: \int^{\text{op}} W \rightarrow \mathcal{C}$ with \mathcal{A} small.

Lax normal conical 2-limit of F :

$$\mathcal{C}\left(U, \text{lax}^n\text{-}\lim^{\Delta^1} F\right) \cong \left[\int^{\text{op}} W, \mathcal{CAT}\right]_{\text{lax}^n}(\Delta 1, \mathcal{C}(U, F(-)))$$

$W: \mathcal{A}^{\text{op}} \rightarrow \mathcal{CAT}$ and $F: \int W \rightarrow \mathcal{C}$ with \mathcal{A} small.

Oplax normal conical 2-colimit of F :

$$\mathcal{C}\left(\text{op}^{\text{lax}^n}\text{-}\text{colim}^{\Delta^1} F, U\right) \cong \left[\left(\int W\right)^{\text{op}}, \mathcal{CAT}\right]_{\text{op}^{\text{lax}^n}}(\Delta 1, \mathcal{C}(F(-), U))$$

Conicalization of weighted 2-limits

Theorem 4 (Street[3], but with a new proof and more detail).

Lax normal conical 2-limits are particular weighted 2-limits, weighted by

$$\begin{aligned} W^{\text{lax}^n} : \int^{\text{op}} Z &\longrightarrow \mathcal{CAT} \\ (B, X') &\mapsto Z(B)/X'. \end{aligned}$$

Conicalization of weighted 2-limits

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$$\begin{aligned} W^{\text{lax}^n} : \int^{\text{op}} Z &\longrightarrow \mathcal{CAT} \\ (B, X') &\mapsto Z(B)/X'. \end{aligned}$$

Given $W: \mathcal{A} \rightarrow \mathcal{CAT}$ with \mathcal{A} small, there is a 2-natural isomorphism

$$\begin{aligned} [W, [y(-), U]] &\cong \left[\int^{\text{op}} W, \mathcal{CAT} \right]_{\text{lax}^n} (\Delta 1, [(y \circ \mathcal{G}(W)^{\text{op}})(-), U]) \\ W &\cong \text{colim}^W y \cong \text{lax}^n\text{-colim}^{\Delta 1} (y \circ \mathcal{G}(W)^{\text{op}}). \end{aligned}$$

Whence every weighted 2-limits is essentially conicalizable:

$$\lim^W F \cong \lim^{W^{\text{lax}^n}} \left(\lim^{(y \circ \mathcal{G}(W)^{\text{op}})(-)} F \right) \cong \text{lax}^n\text{-lim}^{\Delta 1} (F \circ \mathcal{G}(W))$$

Conicalization of weighted 2-limits

Example 5.

$W: \mathcal{A} \rightarrow \mathcal{CAT}$ with \mathcal{A} small.

$$W \cong \operatorname{colim}^W y \cong \operatorname{lax}^n\text{-}\operatorname{colim}^{\Delta^1} (y \circ \mathcal{G}(W)^{\operatorname{op}}).$$

The universal lax normal cocone is given by

$$\forall \begin{array}{c} (A, X) \\ \downarrow (f, \alpha) \\ (A', X') \end{array} \text{ in } \int^{\operatorname{op}} W \qquad \begin{array}{ccc} y(A) & \xrightarrow{[X]} & W \\ y(f) \uparrow & \Downarrow [\alpha] & \uparrow \\ y(A') & \xrightarrow{[X']} & \end{array}$$

Taking $\mathcal{A} = 1$, we obtain that 1 is “lax normal conical dense” in \mathcal{CAT} .

The Grothendieck constr from an abstract point of view

Proposition 6.

$F: \mathcal{B} \rightarrow \mathcal{CAT}$. Consider its 2-*Set*-enriched Grothendieck construction. There is a lax normal natural transformation λ of the form

$$\begin{array}{ccc} \int^{\text{op}} F & \longrightarrow & 1 \\ \mathcal{G}(F) \downarrow & \swarrow \text{lax}^n & \downarrow 1 \\ \mathcal{B} & \xrightarrow{F} & \mathcal{CAT} \end{array}$$

$$\lambda_{(A,X)} = X$$

$$\lambda_{(f,\alpha)} = \alpha$$

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 \mathcal{B} & \xrightarrow{F} & \mathcal{CAT}
 \end{array}
 \quad
 \begin{array}{l}
 \lambda_{(A,X)} = X \\
 \lambda_{(f,\alpha)} = \alpha
 \end{array}$$

Notice that this diagram lives in $2\text{-}\mathcal{CAT}_{\text{lax}}$, that is a lax 3-category (Lambert[2]), where the interchange rule is lax. $2\text{-}\mathcal{CAT}_{\text{lax}} = 2\text{-Set-}\mathcal{CAT}$

$$\begin{array}{ccc}
 \mathcal{A} & \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} & \mathcal{B} & \begin{array}{c} \xrightarrow{H} \\ \Downarrow \beta \\ \xrightarrow{K} \end{array} & \mathcal{C}
 \end{array}
 \quad
 \beta_{\alpha_A}: K(\alpha_A) \circ \beta_{F(A)} \Rightarrow \beta_{G(A)} \circ H(\alpha_A)$$

The Grothendieck constr from an abstract point of view

Definition 7 (lax comma; M., refining Bird[1], Lambert[2]).

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathcal{M} & \xrightarrow{P} & \mathcal{A} \\
 \downarrow Q & \swarrow \forall \gamma & \downarrow F \\
 \mathcal{B} & \xrightarrow{G} & \mathcal{C}
 \end{array} & = & \begin{array}{ccc}
 \mathcal{M} & \xrightarrow{P} & \mathcal{A} \\
 \downarrow Q & \swarrow \exists! V & \downarrow F \\
 \mathcal{B} & \xrightarrow{G} & \mathcal{C}
 \end{array} \\
 \\
 \begin{array}{ccc}
 \mathcal{M} & \xrightarrow{V} & F \parallel G \xrightarrow{\partial_0} \mathcal{A} \\
 \downarrow W & \swarrow \Delta & \downarrow \partial_1 \\
 F \parallel G & \xrightarrow{\partial_1} & \mathcal{B} \xrightarrow{G} \mathcal{C}
 \end{array} & \xRightarrow{\forall \Xi} & \begin{array}{ccc}
 \mathcal{M} & \xrightarrow{V} & F \parallel G \xrightarrow{\partial_0} \mathcal{A} \\
 \downarrow W & \swarrow \Gamma & \downarrow \partial_1 \\
 F \parallel G & \xrightarrow{\partial_1} & \mathcal{B} \xrightarrow{G} \mathcal{C}
 \end{array}
 \end{array}$$

$$\exists! \nu: V \xRightarrow{\text{lax}} W \text{ s.t. } \partial_0 \nu = \Gamma, \partial_1 \nu = \Delta, \lambda_\nu = \Xi.$$

The Grothendieck constr from an abstract point of view

Proposition 8.

$$\begin{array}{ccc}
 \int^{\text{op}} F & \longrightarrow & 1 \\
 \mathcal{G}(F) \downarrow & \swarrow \text{lax comma} & \downarrow 1 \\
 \mathcal{B} & \xrightarrow{F} & \mathcal{CAT}
 \end{array}$$

Or equivalently

$$\begin{array}{ccccc}
 \int^{\text{op}} F & \longrightarrow & \mathcal{CAT}_{\bullet, \text{lax}} & \longrightarrow & 1 \\
 \mathcal{G}(F) \downarrow & \lrcorner & \tau \downarrow & \swarrow \text{lax comma} & \downarrow 1 \\
 \mathcal{B} & \xrightarrow{F} & \mathcal{CAT} & \equiv & \mathcal{CAT}
 \end{array}$$

$$\mathcal{G}(-) : [\mathcal{B}, \mathcal{CAT}] \rightarrow 2\text{-}\mathcal{CAT} / \mathcal{B}.$$

The Grothendieck constr from an abstract point of view

Definition 9.

A **2-Set-opfibration over \mathcal{B}** is a 2-functor $P: \mathcal{E} \rightarrow \mathcal{B}$ such that

- (i) the underlying functor P_0 is an ordinary Grothendieck opfibration;
- (ii) for every pair $X, Y \in \mathcal{E}$ the functor

$$P_{X,Y}: \mathcal{E}(X, Y) \rightarrow \mathcal{B}(P(X), P(Y))$$

is a discrete fibration.

Theorem 10 (Lambert[2]).

The essential image of the 2-functor

$$\mathcal{G}(-): [\mathcal{B}, \mathcal{CAT}] \rightarrow 2\text{-}\mathcal{CAT} / \mathcal{B}$$

is given by the split 2-Set-opfibrations with small fibres.

The Grothendieck constr from an abstract point of view

Theorem 11 (M., the first part is in Bird[1], Street[3]).

$$\begin{aligned} [\mathcal{A}, \mathcal{CAT}]_{\text{lax}}(F, U) &\cong \left[\int^{\text{op}} F, \mathcal{CAT} \right]_{\text{lax}} (\Delta 1, U \circ \mathcal{G}(F)), \\ [\mathcal{A}, \mathcal{CAT}]_{\text{ps}}(F, U) &\cong \left[\int^{\text{op}} F, \mathcal{CAT} \right]_{\text{sigma}} (\Delta 1, U \circ \mathcal{G}(F)) \\ [\mathcal{A}, \mathcal{CAT}](F, U) &\cong \left[\int^{\text{op}} F, \mathcal{CAT} \right]_{\text{lax}^n} (\Delta 1, U \circ \mathcal{G}(F)) \end{aligned}$$

The Grothendieck constr from an abstract point of view

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$$[\mathcal{A}, \mathcal{CAT}]_{\text{lax}}(F, U) \cong \left[\int^{\text{op}} F, \mathcal{CAT} \right]_{\text{lax}}(\Delta 1, U \circ \mathcal{G}(F)),$$

$$[\mathcal{A}, \mathcal{CAT}]_{\text{ps}}(F, U) \cong \left[\int^{\text{op}} F, \mathcal{CAT} \right]_{\text{sigma}}(\Delta 1, U \circ \mathcal{G}(F))$$

$$[\mathcal{A}, \mathcal{CAT}](F, U) \cong \left[\int^{\text{op}} F, \mathcal{CAT} \right]_{\text{lax}^n}(\Delta 1, U \circ \mathcal{G}(F))$$

$$[\mathcal{A}, \mathcal{CAT}]_{\text{lax}}(F, G) \cong \left[\int^{\text{op}} F, \mathcal{CAT} \right]_{\text{lax}}(\Delta 1, G \circ \mathcal{G}(F)) \cong 2\text{-}\mathcal{CAT} / \mathcal{A} \left(\int^{\text{op}} F, \int^{\text{op}} G \right)$$

$$\mathcal{G}(-) : [\mathcal{A}, \mathcal{CAT}]_{\text{lax}} \xrightarrow{\sim} 2\text{-Set-OpFib}(\mathcal{A})$$

$$\mathcal{G}(-) : [\mathcal{A}, \mathcal{CAT}]_{\text{ps}} \xrightarrow{\sim} 2\text{-Set-OpFib}_{\text{cart}}(\mathcal{A})$$

$$\mathcal{G}(-) : [\mathcal{A}, \mathcal{CAT}] \xrightarrow{\sim} 2\text{-Set-OpFib}_{\text{clop}}(\mathcal{A})$$

A pointwise Kan extension result

Definition 12 (colimits in $2\text{-}\mathcal{CAT}_{\text{lax}}$; \mathbf{M}).

$M: \mathcal{A}^{\text{op}} \rightarrow \mathcal{CAT}$ (the marking) with \mathcal{A} small,

$F: \int M \rightarrow \mathcal{C}$ (the diagram), $W: (\int M)^{\text{op}} \rightarrow \mathcal{CAT}$ (the weight).

Oplax normal 2-colimit of F marked by M and weighted by W :

$$\mathcal{C} \left(\text{op} \text{ lax } {}^n_M\text{-colim } {}^W F, U \right) \cong \left[\left(\int M \right)^{\text{op}}, \mathcal{CAT} \right]_{\text{op} \text{ lax } {}^n} (W, \mathcal{C}(F(-), U))$$

We say **opmarked** when the domain of F is expressed as $\int^{\text{op}} M$ for some $M: \mathcal{A} \rightarrow \mathcal{CAT}$.

Theorem 13.

In $2\text{-}\mathcal{CAT}_{\text{lax}}$ every trivially marked weighted 2-colimit can be equivalently expressed as a marked trivially weighted 2-colimit.

A pointwise Kan extension result

Definition 14 (M.).

$$\begin{array}{ccc}
 \mathcal{B} & \xrightarrow{F} & \mathcal{C} \\
 K \downarrow & \swarrow \lambda \text{ lax}^n & \nearrow \text{Lan}_K F \\
 \mathcal{A} & &
 \end{array}$$

with \mathcal{B} small and $K = \mathcal{G}(M)$ a 2-Set-opfibration.

Pointwise left Kan extension of F along K :

$$(\text{Lan}_K F)(A) \cong \text{oplax}^{\text{op-n}}_M\text{-colim}^{\mathcal{A}(K(-), A)} F$$

with universal oplax normal cocylinder

$$\begin{aligned}
 \mathcal{A}(K(-), A) &\xRightarrow{L} \mathcal{C}((L \circ K)(-), L(A)) \xrightarrow[\text{oplax}^n]{C(\lambda_-, \text{id})} \mathcal{C}(F(-), L(A)) \\
 \mathcal{C}(L(A), C) &\cong [\mathcal{B}^{\text{op}}, \mathcal{CAT}]_{\text{oplax}^n}(\mathcal{A}(K(-), A), \mathcal{C}(F(-), C))
 \end{aligned}$$

A pointwise Kan extension result

Theorem 15 (M.).

$F: \mathcal{A} \rightarrow \mathcal{CAT}$ with \mathcal{A} small. Then the 2-*Set*-enriched Grothendieck construction lax comma square exhibits

$$F = \mathrm{Lan}_{g(F)} \Delta 1.$$

Proof.

$$\begin{aligned} \mu_{(B,X)} : \mathcal{B}(B,A) &\longrightarrow \mathcal{CAT}(1, F(A)) \\ (B \xrightarrow{u} A) &\longmapsto F(u)(X) \\ (\mu_{(g,\gamma)})_u = F(u)(\gamma) : F(u \circ g)(X') &\rightarrow F(u)(X) \end{aligned}$$

is 2-universal:

A pointwise Kan extension result

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Proof. $\mu_{(B,X)}: \mathcal{B}(B,A) \longrightarrow \mathcal{CAT}(1, F(A))$

$$(B \xrightarrow{u} A) \longmapsto F(u)(X)$$

$$(\mu_{(g,\gamma)})_u = F(u)(\gamma): F(u \circ g)(X') \rightarrow F(u)(X)$$

is 2-universal: given $Q: \mathcal{A}(\mathcal{G}(F)(-), A) \xRightarrow[\mathrm{oplax}^n]{} \mathcal{CAT}(\Delta 1(-), C)$,

$\exists! s: F(A) \rightarrow C$ such that $(s \circ -) \circ \mu = Q$.

$$s(X) = s(\mu_{(A,X)}(\mathrm{id}_A)) = Q_{(A,X)}(\mathrm{id}_A)$$

$$s(\alpha) = s\left((\mu_{(\mathrm{id}_A, \alpha)})_{\mathrm{id}_A}\right) = (Q_{(\mathrm{id}_A, \alpha)})_{\mathrm{id}_A}.$$

A pointwise Kan extension result

Definition 16 (M.).

$G, H: \mathcal{B}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{E}$. **Oplaxⁿ-lax natural transformation** $\alpha: G \Rightarrow H$:

$$\alpha_{B,C}: G(B, C) \rightarrow H(B, C)$$

$\alpha_{-,C}$ oplax normal in B , $\alpha_{B,-}$ lax in C and compatibility

$$\begin{array}{ccc}
 G(B, C) & \xrightarrow{\alpha_{B,C}} & H(B, C) \\
 \uparrow G(f, \text{id}) & & \downarrow H(f, \text{id}) \\
 G(B', C) & \xrightarrow{\alpha_{B,C'}} & H(B, C') \\
 \downarrow G(\text{id}, g) & & \downarrow H(\text{id}, g) \\
 G(B', C') & \xrightarrow{\alpha_{B',C'}} & H(B', C')
 \end{array}
 =
 \begin{array}{ccc}
 G(B, C) & \xrightarrow{\alpha_{B,C}} & H(B, C) \\
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 \downarrow G(\text{id}, g) & & \downarrow H(\text{id}, g) \\
 G(B', C') & \xrightarrow{\alpha_{B',C'}} & H(B', C')
 \end{array}$$

Modifications between them are modifications in each variable.

A pointwise Kan extension result

Theorem 17 (The oplaxⁿ-lax param Yoneda lemma; M.).

$K: \mathcal{B} \rightarrow \mathcal{A}$ a 2-Set-opfibration, $F: \mathcal{B}^{\text{op}} \times \mathcal{A} \rightarrow \mathcal{CAT}$.

There is a bijection between

$$\alpha_{B,A}: \mathcal{A}(K(B), A) \rightarrow F(B, A)$$

oplaxⁿ-lax natural in $(B, A) \in \mathcal{B}^{\text{op}} \times \mathcal{A}$ and

$$\eta_B: 1 \rightarrow F(B, K(B))$$

extraordinary lax natural in $B \in \mathcal{B}$.

Moreover this bijection extends to an isomorphism of categories.

A pointwise Kan extension result

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$$\eta_B: 1 \rightarrow F(B, K(B))$$

extraordinary lax natural in $B \in \mathcal{B}$.

Moreover this bijection extends to an isomorphism of categories.

Corollary 18 (M.).

Every pointwise left Kan extension in $2\text{-}\mathcal{CAT}_{\text{lax}}$ is also a weak one.



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