Derived categories and Maximal Cohen-Macaulay modules

Reading Course in Commutative Algebra

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Part I

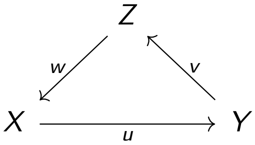
Triangulated categories

Triangles

Definition 1.1.

Let $\mathfrak{C}$ be a category and let T be an automorphism. A **triangle** of $\mathfrak{C}$ is a sextuple (X, Y, Z, u, v, w) where X, Y, Z are objects of $\mathfrak{C}$ and $u\colon X \to Y$, $v\colon Y \to Z$, $w\colon Z \to T(X)$ are morphisms of $\mathfrak{C}$.

A triangle is usually written as follows:



or even

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X).$$

Triangles

Definition 1.2.

Let (X, Y, Z, u, v, w) and $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{u}, \tilde{v}, \tilde{w})$ be two triangles of $\mathfrak{C}$. A **morphism of triangles**

$$(X, Y, Z, u, v, w) \to (\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{u}, \tilde{v}, \tilde{w})$$

is a triple of morphisms (f, g, h) forming a commutative diagram in $\mathfrak{C}$:

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\ \tilde{X} & \xrightarrow{\tilde{u}} & \tilde{Y} & \xrightarrow{\tilde{v}} & \tilde{Z} & \xrightarrow{\tilde{w}} & T(\tilde{X}). \end{array}$$

Triangulated categories

Definition 1.3.

An additive category $\mathfrak{C}$ is a **triangulated category** if it is equipped with

- an authomorfism $T\colon \mathfrak{C} \to \mathfrak{C}$, called the **translation functor**, and
- a collection of triangles (X, Y, Z, u, v, w), called the **exact triangles** (or **distinguished triangles**) of $\mathfrak{C}$,

such that the following axioms hold:

TR1:

- For each X object of $\mathfrak{C}$, the triangle $(X, X, 0, \mathrm{id}, 0, 0)$ is exact.
- Every morphism $u : X \to Y$ can be embedded in an exact triangle (X, Y, Z, u, v, w).
- If (X, Y, Z, u, v, w) is a triangle isomorphic to an exact triangle $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{u}, \tilde{v}, \tilde{w})$, i.e.

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
X & \xrightarrow{u} & Y & \xrightarrow{\tilde{v}} & \tilde{Z} & \xrightarrow{\tilde{w}} & T(X),
\end{array}$$

then (X, Y, Z, u, v, w) is also exact.

Triangulated categories

TR2: [Rotation] If (X, Y, Z, u, v, w) is an exact triangle, then $(Y, Z, T(X), v, w, -T(u))$ and $(T^{-1}(Z), X, Y, -T^{-1}(w), u, v)$ are exact triangles. Equivalently, (X, Y, Z, u, v, w) is an exact triangle if and only if $(Y, Z, T(X), v, w, -T(u))$ is so.

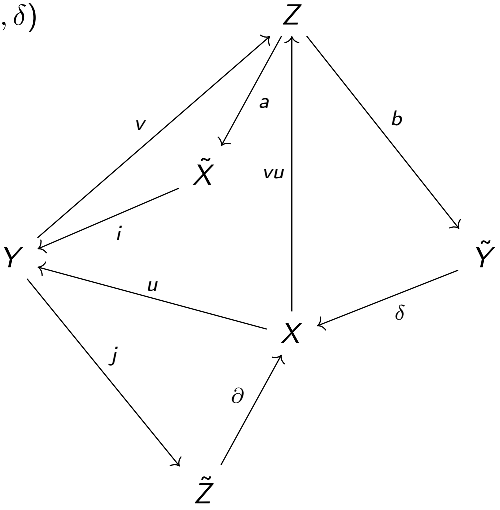
TR3: [Morphisms] Given two exact triangles (X, Y, Z, u, v, w) and $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{u}, \tilde{v}, \tilde{w})$ with morphisms $f\colon X \to \tilde{X}$, $g\colon Y \to \tilde{Y}$ such that $gu = \tilde{u}f$, then there exists a morphism $h\colon Z \to \tilde{Z}$ such that (f, g, h) is a morphism of triangles, i.e. the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\ \tilde{X} & \xrightarrow{\tilde{u}} & \tilde{Y} & \xrightarrow{\tilde{v}} & \tilde{Z} & \xrightarrow{\tilde{w}} & T(\tilde{X}) \end{array}$$

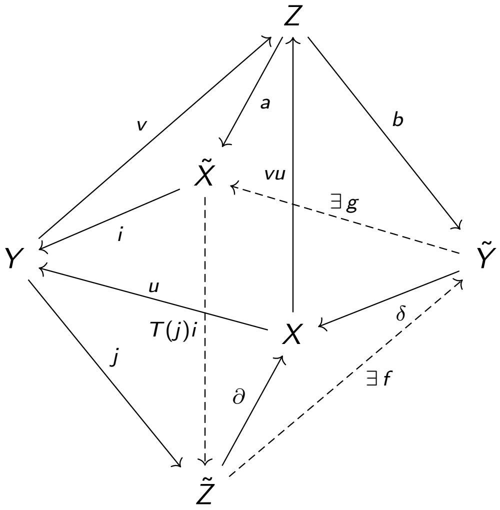
is commutative.

Triangulated categories

TR4: [The octahedral axiom] Given exact triangles $(X, Y, \tilde{Z}, u, j, \partial)$, $(Y, Z, \tilde{X}, v, a, i)$ and $(X, Z, \tilde{Y}, vu, b, \delta)$



there exist morphisms $f\colon \tilde{Z} \to \tilde{Y}$, $g\colon \tilde{Y} \to \tilde{X}$ such that $(\tilde{Z}, \tilde{Y}, \tilde{X}, f, g, T(j)i)$ is an exact triangle and such that in the following octahedron



Triangulated categories

we have:

i) the four exact triangles form four of the faces;
ii) the remaining four faces commute, i.e. $\partial = \delta f : \tilde{Z} \to X$ and $a = gb \colon Z \to \tilde{X}$;
iii) $bv = fj \colon Y \to \tilde{Y}$;
iv) $ig = u\delta \colon \tilde{Y} \to Y$.

Remark 1.4.

From the axioms required to a triangulated category it follows rather easily that

- If (X, Y, Z, u, v, w) is an exact triangle, then

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X).$$

  is a complex.
- If $\mathfrak{C}$ is a triangulated category, then $\mathfrak{C}^{op}$ is a triangulated category.

Triangulated functors

Definition 1.5.

Let $\mathfrak{C}$ and $\tilde{\mathfrak{C}}$ be two triangulated categories. An additive functor $F\colon \mathfrak{C} \to \tilde{\mathfrak{C}}$ is called a **covariant (strict) triangulated functor** if it commutes with the translation functor and takes exact triangles into exact triangles.
F is called a **contravariant (strict) triangulated functor** if it takes exact triangles into exact triangles with the arrows reversed and it sends the translation functor into its inverse.

Cohomological functors

Definition 1.6.

Let $(\mathfrak{C}, T)$ be a triangulated category and $\mathfrak{A}$ be an abelian category. An additive functor $H\colon \mathfrak{C} \to \mathfrak{A}$ is called a **covariant cohomological functor** if whenever (X, Y, Z, u, v, w) is an exact triangle, the long sequence

$$\cdots \to H(T^i(X)) \xrightarrow{H(T^i(u))} H(T^i(Y)) \xrightarrow{H(T^i(v))} H(T^i(Z)) \xrightarrow{H(T^i(w))} H(T^{i+1}(X)) \xrightarrow{H(T^{i+1}(u))} \cdots$$

is exact in $\mathfrak{A}$. We often write $H^i(-)$ for $H(T^i(-))$, $i \in \mathbb{Z}$.

One defines a **contravariant cohomological functor** by reversing the arrows.

Example 1.7.

Let $\mathfrak{C}$ be a triangulated category and M an object of $\mathfrak{C}$. Then $\mathrm{Hom}_{\mathfrak{C}}(M,-)$ and $\mathrm{Hom}_{\mathfrak{C}}(-,M)$ are cohomological functors on $\mathfrak{C}$.

Let (X,Y,Z,u,v,w) be an exact triangle. First we show that

$$\mathrm{Hom}_{\mathfrak{C}}(M,X) \xrightarrow{\mathrm{Hom}_{\mathfrak{C}}(M,u)} \mathrm{Hom}_{\mathfrak{C}}(M,Y) \xrightarrow{\mathrm{Hom}_{\mathfrak{C}}(M,v)} \mathrm{Hom}_{\mathfrak{C}}(M,Z)$$

is exact. Since $\mathrm{Hom}_{\mathfrak{C}}(M,u) = u \circ -$ and $\mathrm{Hom}_{\mathfrak{C}}(M,v) = v \circ -$, by Remark 1.4 we have that

$$\mathrm{Hom}_{\mathfrak{C}}(M,v)(\mathrm{Hom}_{\mathfrak{C}}(M,u)(f)) = (vu)f = 0 \quad \text{for each } f \in \mathrm{Hom}_{\mathfrak{C}}(M,X).$$

Cohomological functors

Suppose now that $g \in \mathrm{Hom}_{\mathfrak{C}}(M, Y)$ is such that $vg = 0$. Applying TR2 and TR3 to the exact triangles $(M, M, 0, \mathrm{id}_M, 0, 0)$ and (X, Y, Z, u, v, w) we have the commutative diagram

$$\begin{array}{ccccccc}
M & \xrightarrow{0} & 0 & \xrightarrow{0} & T(M) & \xrightarrow{-T(\mathrm{id}_M)} & T(M) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle T(g)} \\
Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) & \xrightarrow{-T(u)} & T(Y).
\end{array}$$

Hence $g = T^{-1}(T(g)) = T^{-1}(T(u)h) = uT^{-1}(h)$. We conclude that there exists a $f = T^{-1}(h) \in \mathrm{Hom}_{\mathfrak{C}}(M, X)$ such that $g = uf$. By TR2 $(Y, Z, T(X), v, w, -T(u))$ and $(T^{-1}(Z), X, Y, -T^{-1}(w), u, v)$ are exact triangles.

Thus

$$\mathrm{Hom}_{\mathfrak{C}}(M,Y) \xrightarrow{\mathrm{Hom}_{\mathfrak{C}}(M,v)} \mathrm{Hom}_{\mathfrak{C}}(M,Z) \xrightarrow{\mathrm{Hom}_{\mathfrak{C}}(M,w)} \mathrm{Hom}_{\mathfrak{C}}(M,T(X))$$

and

$$\mathrm{Hom}_{\mathfrak{C}}(M,T^{-1}(Z)) \xrightarrow{\mathrm{Hom}_{\mathfrak{C}}(M,-T^{-1}(w))} \mathrm{Hom}_{\mathfrak{C}}(M,X) \xrightarrow{\mathrm{Hom}_{\mathfrak{C}}(M,u)} \mathrm{Hom}_{\mathfrak{C}}(M,Y)$$

are exact. Using TR2 we obtain that $\mathrm{Hom}_{\mathfrak{C}}(M,-)$ is a covariant cohomological functor.

A similar proof shows that $\mathrm{Hom}_{\mathfrak{C}}(-,M)$ is a contravariant cohomological functor.

Triangulated Five lemma

Proposition 1.8.

In a triangulated category, in the situation of TR3

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow T(f) \\ \tilde{X} & \xrightarrow{\tilde{u}} & \tilde{Y} & \xrightarrow{\tilde{v}} & \tilde{Z} & \xrightarrow{\tilde{w}} & T(\tilde{X}) \end{array}$$

if f and g are isomorphisms, then h is also an isomorphism.

Suppose that f and g are isomorphisms. Applying $\mathrm{Hom}_{\mathfrak{C}}(\tilde{Z}, -)$, which is cohomological by Example 1.7, we obtain a commutative diagram with exact rows

Triangulated Five lemma

$$\begin{array}{ccccccccc}
\mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},X) & \to & \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},Y) & \to & \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},Z) & \to & \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},T(X)) & \to & \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},T(Y)) \\
\downarrow {\scriptstyle \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},f)} & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},g)} & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},h)} & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},T(f))} & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},T(g))} \\
\mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},\tilde{X}) & \to & \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},\tilde{Y}) & \to & \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},\tilde{Z}) & \to & \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},T(\tilde{X})) & \to & \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},T(\tilde{Y})).
\end{array}$$

Since f, g, $T(f)$ and $T(g)$ are isomorphisms, $\mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},f)$, $\mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},g)$, $\mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},T(f))$ and $\mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},T(g))$ are isomorphisms of abelian groups.

Hence by the Five lemma, $\mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},h)$ is an isomorphism. Therefore, there exists $\varphi \in \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},Z)$ such that

$$\mathrm{id}_{\tilde{Z}} = \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z},h)(\varphi) = h\varphi.$$

Triangulated Five lemma

Similarly, by applying $\mathrm{Hom}_{\mathfrak{C}}(-, Z)$ we have that there exists $\psi \in \mathrm{Hom}_{\mathfrak{C}}(\tilde{Z}, Z)$ such that

$$\mathrm{id}_Z = \mathrm{Hom}_{\mathfrak{C}}(h, Z)(\psi) = \psi h.$$

Thus $\varphi = \mathrm{id}_Z \varphi = \psi h \varphi = \psi \, \mathrm{id}_{\tilde{Z}} = \psi$, and so h is an isomorphism.

Corollary 1.9.

Every exact triangle is uniquely determined up to isomorphism by any one of its maps.

Idea:

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\
\downarrow{\scriptstyle \mathrm{id}_X} & & \downarrow{\scriptstyle \mathrm{id}_Y} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{id}_{T(X)}} \\
\tilde{X} & \xrightarrow{u} & \tilde{Y} & \xrightarrow{\tilde{v}} & \tilde{Z} & \xrightarrow{\tilde{w}} & T(\tilde{X}).
\end{array}$$

Homotopy category

Definition 1.10.

Let $\mathfrak{A}$ be an abelian category and let $\mathcal{CoCh}(\mathfrak{A})$ be the category of cochain complexes in $\mathfrak{A}$. We define the **homotopy category** $\mathcal{K}(\mathfrak{A})$ of $\mathfrak{A}$ to be the category whose objects are cochain complexes (the objects of $\mathcal{CoCh}(\mathfrak{A})$) and whose morphisms are homotopy equivalence classes of morphisms in $\mathcal{CoCh}(\mathfrak{A})$ (i.e. $\mathcal{K}(\mathfrak{A})$ is the quotient category of $\mathcal{CoCh}(\mathfrak{A})$ by the chain homotopy equivalence).

Homotopy category

Remark 1.11.

$\mathcal{K}(\mathfrak{A})$ is an additive category and the quotient functor $\mathcal{C}o\mathcal{C}h(\mathfrak{A}) \to \mathcal{K}(\mathfrak{A})$ is an additive functor. (But in general $\mathcal{K}(\mathfrak{A})$ is not an abelian category.)

Moreover $\mathcal{K}(\mathfrak{A})$ has a universal property: if $F\colon \mathcal{C}o\mathcal{C}h(\mathfrak{A}) \to \mathfrak{D}$ is a functor which sends homotopy equivalences to isomorphisms, then F factors uniquely through $\mathcal{K}(\mathfrak{A})$.

$$\begin{array}{ccc} \mathcal{C}o\mathcal{C}h(\mathfrak{A}) & \xrightarrow{F} & \mathfrak{D} \\ \downarrow & \nearrow_{\exists!} & \\ \mathcal{K}(\mathfrak{A}) & & \end{array}$$

Bounded complexes

Recall 1.12.

A complex $X^\bullet$ is said to be **bounded below** if there exists $n \in \mathbb{Z}$ such that $X^i = 0$ for all $i < n$; $X^\bullet$ is said to be **bounded above** if there exists $n \in \mathbb{Z}$ such that $X^i = 0$ for all $i > n$; $X^\bullet$ is said to be **bounded (on both sides)** if it is bounded below and bounded above.

We denote $\mathcal{K}^+(\mathfrak{A})$ (respectively $\mathcal{K}^-(\mathfrak{A})$, $\mathcal{K}^{\mathrm{b}}(\mathfrak{A})$) the full subcategory of $\mathcal{K}(\mathfrak{A})$ consisting of the complexes bounded below (respectively bounded above, bounded on both sides).

The triangulated structure on $\mathcal{K}(\mathfrak{A})$

Construction 1.13.

We define now a structure of triangulated category on $\mathcal{K}(\mathfrak{A})$. As translation functor we take T the functor which shifts complexes one place to the left and changes the sign of the differential, i.e.

$$T(X^\bullet)^i = X^{i+1} \quad \text{and} \quad \mathrm{d}^i_{T(X^\bullet)} = -\,\mathrm{d}^{i+1}_X\,.$$

We will often write $X^\bullet[1]$ instead of $T(X^\bullet)$, and $X^\bullet[n]$ instead of $T^n(X^\bullet)$.

Let $u\colon X^\bullet \to Y^\bullet$ be a morphism in $\mathcal{K}(\mathfrak{A})$. The **mapping cone** of u is the cochain complex $T(X^\bullet) \oplus Y^\bullet$, where the differential operator is given by

$$\mathrm{d}^i = \begin{bmatrix} \mathrm{d}^i_{T(X^\bullet)} & 0 \\ T(u) & \mathrm{d}^i_{Y^\bullet} \end{bmatrix} = \begin{bmatrix} -\,\mathrm{d}^{i+1}_{X^\bullet} & 0 \\ u^{i+1} & \mathrm{d}^i_{Y^\bullet} \end{bmatrix}.$$

The triangulated structure on $\mathcal{K}(\mathfrak{A})$

By construction, we have that u can be completed to a triangle (which is unique up to isomorphism by Corollary 1.9). We define an exact triangle in $\mathcal{K}(\mathfrak{A})$ to be any triangle isomorphic to a triangle given by the mapping cone of a morphism of complexes, i.e. if there exists $u : X^\bullet \to Y^\bullet$ and a commutative diagram in $\mathcal{K}(\mathfrak{A})$

$$\begin{array}{ccccccc}
X^\bullet & \xrightarrow{u} & Y^\bullet & \xrightarrow{v} & T(X^\bullet) \oplus Y^\bullet & \xrightarrow{w} & T(X^\bullet) \\
{\scriptstyle f}\uparrow & & {\scriptstyle g}\uparrow & & {\scriptstyle h}\uparrow & & {\scriptstyle T(f)}\uparrow \\
\tilde{X}^\bullet & \xrightarrow{\tilde{u}} & \tilde{Y}^\bullet & \xrightarrow{\tilde{v}} & \tilde{Z}^\bullet & \xrightarrow{\tilde{w}} & T(\tilde{X}^\bullet)
\end{array}$$

such that f, g, h are isomorphisms in $\mathcal{K}(\mathfrak{A})$, where $v\colon Y^\bullet \to T(X^\bullet) \oplus Y^\bullet$ and $w\colon T(X^\bullet) \oplus Y^\bullet \to T(X^\bullet)$ are the natural maps.

Proposition 1.14.

$\mathcal{K}(\mathfrak{A})$ is a triangulated category.

Cohomology functor

Proposition 1.15.

Let $\mathfrak{A}$ be an abelian category. The functor

$$\begin{aligned} H\colon\ \mathcal{K}(\mathfrak{A}) &\longrightarrow \mathfrak{A} \\ X^\bullet &\longmapsto \ker d^0_{X^\bullet} \,/\, \operatorname{Im} d^{-1}_{X^\bullet} \end{aligned}$$

is a cohomological functor.

H is additive. Let $(X^\bullet, Y^\bullet, Z^\bullet, u, v, w)$ be an exact triangle in $\mathcal{K}(\mathfrak{A})$ with $Z^\bullet = T(X^\bullet) \oplus Y^\bullet$ the mapping cone of u. Then $Y^\bullet \to Z^\bullet \to T(X^\bullet)$ is split exact and so

$$H^i(Y^\bullet) \to H^i(Z^\bullet) \to H^i(T(X^\bullet))$$

remains exact. By TR2 we are done.

Triangulated subcategories

Definition 1.16.

Let $\mathfrak{C}$ be a triangulated category. A full additive subcategory $\mathfrak{D}$ of $\mathfrak{C}$ is said a **triangulated subcategory** if the following conditions hold:

- Each object isomorphic to an object of $\mathfrak{D}$ is in $\mathfrak{D}$;
- If T is the translation functor of $\mathfrak{C}$, then $T(X) \in \mathrm{Ob}(\mathfrak{D})$ for all $X \in \mathrm{Ob}(\mathfrak{D})$.
- If (X, Y, Z, u, v, w) is an exact triangle in $\mathfrak{C}$ such that two of the three objects X, Y, Z belong to $\mathfrak{D}$, then (X, Y, Z, u, v, w) is all contained in $\mathfrak{D}$.

Example 1.17.

Let $\mathfrak{A}$ be an abelian category. $\mathcal{K}^{+}(\mathfrak{A})$, $\mathcal{K}^{-}(\mathfrak{A})$ and $\mathcal{K}^{\mathrm{b}}(\mathfrak{A})$ are triangulated subcategories of $\mathcal{K}(\mathfrak{A})$.

Part II

Localization of categories

Multiplicative systems

Definition 2.1.

Let $\mathfrak{C}$ be a category. A collection S of morphisms of $\mathfrak{C}$ is called a **multiplicative system** if it satisfies the following three axioms:

MS1:
- $\mathrm{id}_X \in S$ for each $X \in \mathrm{Ob}(\mathfrak{C})$;
- If $f, g \in S$ and fg exists, then $fg \in S$.

MS2:
- If $s\colon Z \to Y$ is in S, then for every $u\colon X \to Y$ in $\mathfrak{C}$ there is a commutative diagram $(sv = ut)$ in $\mathfrak{C}$ with $t \in S$:

$$\begin{array}{ccc} W & \overset{v}{\dashrightarrow} & Z \\ {\scriptstyle t}\,\downarrow & & \downarrow\,{\scriptstyle s} \\ X & \underset{u}{\longrightarrow} & Y; \end{array}$$

Multiplicative systems

- If $\tilde{s}\colon Y \to Z$ is in S, then for every $\tilde{u}\colon Y \to X$ in $\mathfrak{C}$ there is a commutative diagram ($\tilde{v}\tilde{s} = \tilde{t}\tilde{u}$) in $\mathfrak{C}$ with $\tilde{t} \in S$:

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{u}} & X \\ {\scriptstyle \tilde{s}}\downarrow & & \downarrow{\scriptstyle \tilde{t}} \\ Z & \xrightarrow[\tilde{v}]{} & W; \end{array}$$

MS3: If $f, g\colon X \to Y$ are morphisms in $\mathfrak{C}$, then the following two conditions are equivalent:

- there exists a morphism $s\colon Y \to Y'$ in S such that $sf = sg$;
- there exists a morphism $t\colon X' \to X$ in S such that $ft = gt$.

Localization of categories

Definition 2.2.

Let $\mathfrak{C}$ be a category and let S be a multiplicative system in $\mathfrak{C}$. A localization of $\mathfrak{C}$ with respect to S is a category $\mathfrak{C}_S$ together with a functor $Q\colon \mathfrak{C} \to \mathfrak{C}_S$ such that

(a) $Q(s)$ is an isomorphism for every $s \in S$, and

(b) for each category $\mathfrak{D}$ and for each functor $F\colon \mathfrak{C} \to \mathfrak{D}$ such that $F(s)$ is an isomorphism for all $s \in S$, there exists a unique functor $F_S\colon \mathfrak{C}_S \to \mathfrak{D}$ such that $F = F_S \circ Q$.

We observe that condition (b), which says that $(\mathfrak{C}_S, Q)$ has a universal property, implies that the localization of $\mathfrak{C}$ with respect to S is unique up to isomorphisms.

The construction of a localization

Construction 2.3.

We now want to find a way to construct the localization of a category.

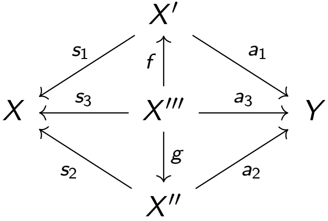
Let $\mathfrak{C}$ be a category and let S be a multiplicative system in $\mathfrak{C}$. We call a diagram in $\mathfrak{C}$ of the form

$$X \xleftarrow{\ s\ } X' \xrightarrow{\ a\ } Y$$

with $s \in S$ a **left fraction** in $\mathfrak{C}$, and we denote it (s, a).

The construction of a localization

We define a relation on the fractions: for each two left fractions (s_1, a_1) and (s_2, a_2), $(s_1, a_1) \sim (s_2, a_2)$ if there exists a left fraction (s_3, a_3) and there exist $f\colon X''' \to X'$, $g\colon X''' \to X''$ morphisms in $\mathfrak{C}$ such that the following diagram



is commutative in $\mathfrak{C}$ (i.e. $s_1 f = s_3 = s_2 g$ and $a_1 f = a_3 = a_2 g$). One can prove that $\sim$ is an equivalence relation. We denote by as^{-1} the equivalence class of (s, a).

The construction of a localization

We obtain a new category $\tilde{\mathfrak{C}}$ setting:

- $\mathrm{Ob}(\tilde{\mathfrak{C}}) = \mathrm{Ob}(\mathfrak{C})$;
- If X and Y are in $\tilde{\mathfrak{C}}$, the morphisms $X \to Y$ in $\tilde{\mathfrak{C}}$ are equivalence classes as^{-1} of fractions as above;
- The composition of two equivalence classes $as^{-1} \colon X \to X' \to Y$ and $bt^{-1} \colon Y \to Y' \to Z$ is defined as follows:
  since we have the diagram

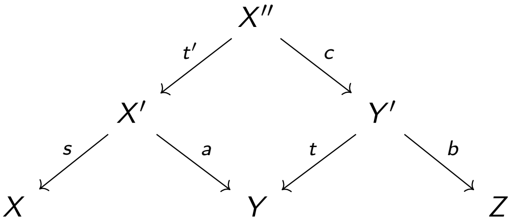
$$X \xleftarrow{s} X' \xrightarrow{a} Y \xleftarrow{t} Y' \xrightarrow{b} Z$$

  by MS2 there exists

$$X' \xleftarrow{t'} X'' \xrightarrow{c} Y'$$

The construction of a localization

with $t' \in S$ such that the diagram



is commutative. Thus we define the composition as

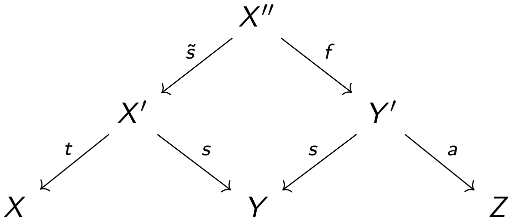
$$bt^{-1} \circ as^{-1} := bc(st')^{-1} \colon X \to Z.$$

One can prove that the composition is well defined, i.e. if $a_1 s_1^{-1} = a_2 s_2^{-1}$ and $b_1 t_1^{-1} = b_2 t_2^{-1}$, then $b_1 t_1^{-1} \circ a_1 s_1^{-1} = b_2 t_2^{-1} \circ a_2 s_2^{-1}$.

The construction of a localization

Remark 2.4.

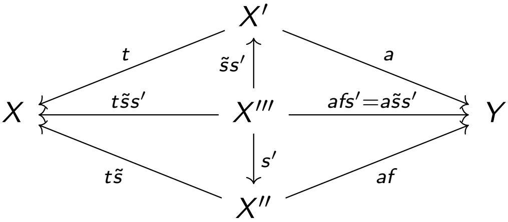
Considering as^{-1} and st^{-1} morphisms in $\tilde{\mathfrak{C}}$, by MS2 we have the commutative diagram



with $\tilde{s} \in S$. Hence $as^{-1} \circ st^{-1} = (af)(t\tilde{s})^{-1}$ and $s\tilde{s} = sf$. By MS3, there exists a morphism $s' \colon W \to X''$ in S such that $fs' = \tilde{s}s' \in S$.

The construction of a localization

Therefore the diagram



is commutative in $\mathfrak{C}$. Then

$$as^{-1} \circ st^{-1} = (af)(t\tilde{s})^{-1} = at^{-1}.$$

The construction of a localization

Proposition 2.5.

Let $\mathfrak{C}$ be a category, S a multiplicative system in $\mathfrak{C}$ and let $\tilde{\mathfrak{C}}$ be the category obtained as above.

Let P be the canonical functor $\mathfrak{C} \to \tilde{\mathfrak{C}}$ which is the identity map on objects and which takes a morphism $a\colon X \to Y$ to $a(\mathrm{id}_X)^{-1}$:

$$\begin{array}{c} X \\ \Big\downarrow a \\ Y \end{array} \quad \overset{P}{\longmapsto} \quad \begin{array}{ccc} X & & \\ & \nwarrow \mathrm{id}_X & \\ & & X \\ & \swarrow a & \\ Y & & \end{array}$$

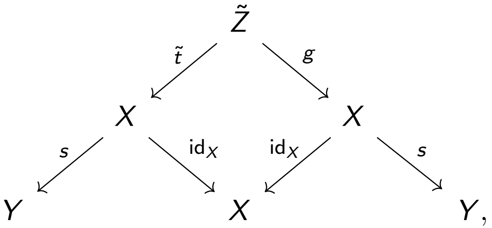
Then $(\tilde{\mathfrak{C}}, P)$ is the localization of $\mathfrak{C}$ with respect to S.

The construction of a localization

(a): Let $s\colon X \to Y$ be in S. We show that $P(s) = s(\mathrm{id}_X)^{-1}$ is an isomorphism. By Remark 2.4,

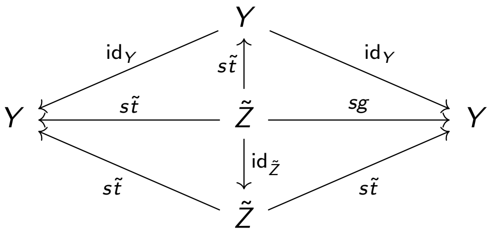
$$\mathrm{id}_X\, s^{-1} \circ P(s) = \mathrm{id}_X\, s^{-1} \circ s(\mathrm{id}_X)^{-1} = \mathrm{id}_X\, \mathrm{id}_X^{-1},$$

and using the commutative diagrams



whence $\tilde{t} = g$, and

The construction of a localization



we have $P(s) \circ \mathrm{id}_X\, s^{-1} = s(\mathrm{id}_X)^{-1} \circ \mathrm{id}_X\, s^{-1} = \mathrm{id}_Y\, \mathrm{id}_Y^{-1}$.

(b): Suppose that $F\colon \mathfrak{C} \to \mathfrak{D}$ is a functor such that $F(s)$ is an isomorphism for all $s \in S$. We define a functor $F_S\colon \tilde{\mathfrak{C}} \to \mathfrak{D}$ as follows:

$$F_S(X) := F(X) \quad \text{for each } X \text{ in } \tilde{\mathfrak{C}} \quad \text{and}$$
$$F_S(as^{-1}) := F(a)F(s)^{-1} \quad \text{for each } as^{-1}\colon X \to Y \text{ in } \tilde{\mathfrak{C}}.$$

Hence for each $a\colon X \to Y$ in $\mathfrak{C}$

$$F_S \circ P(X \xrightarrow{a} Y) = F_S(X \xrightarrow{a(\mathrm{id}_X)^{-1}} Y) = F(X) \xrightarrow{F(a)} F(Y) = F(X \xrightarrow{a} Y).$$

The construction of a localization

If $\tilde{F}_S \colon \tilde{\mathfrak{C}} \to \mathfrak{D}$ is a functor such that $\tilde{F}_S \circ P(X' \xrightarrow{a} Y) = F(X' \xrightarrow{a} Y)$ for each $a \colon X' \to Y$ in $\mathfrak{C}$, then by Remark 2.4

$$\tilde{F}_S(a(\mathrm{id}_{X'})^{-1}) = \tilde{F}_S(as^{-1} \circ s(\mathrm{id}_{X'})^{-1}) = \tilde{F}_S(as^{-1})\tilde{F}_S(s(\mathrm{id}_{X'})^{-1}) =$$
$$= \tilde{F}_S(as^{-1})(\tilde{F}_S \circ P(X' \xrightarrow{s} X)) = \tilde{F}_S(as^{-1})F(s).$$

On the other hand,

$$\tilde{F}_S(a(\mathrm{id}_{X'})^{-1}) = \tilde{F}_S \circ P(X' \xrightarrow{a} Y) = F(a),$$

and so $\tilde{F}_S(as^{-1}) = F(a)F(s)^{-1} = F_S(as^{-1})$ for each as^{-1} morphism in $\tilde{\mathfrak{C}}$, i.e. $\tilde{F}_S = F_S$.

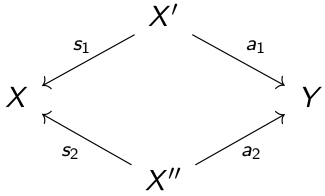
The triangulated structure on the localization

Proposition 2.6.

Let $\mathfrak{C}$ be a category and let S be a multiplicative system in $\mathfrak{C}$. If $\mathfrak{C}$ is an additive category then so is $(\mathfrak{C}_S, Q)$ and Q is an additive functor.

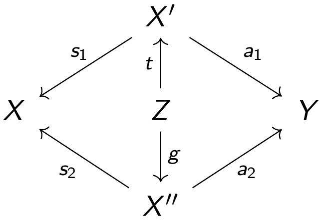
Assume $\mathfrak{C}$ is an additive category. It is easy to check that Q sends an initial object (respectively terminal object) to an initial object (respectively terminal object) and also binary products to binary products. Thus $\mathfrak{C}_S$ has zero object and binary products.

Let now X and Y be two objects in $\mathfrak{C}_S$, and let $a_1 s_1^{-1}, a_2 s_2^{-1} \colon X \to Y$ be two morphisms in $\mathfrak{C}_S$, i.e. we have the diagram



The triangulated structure on the localization

By MS2 there exist $t \in S$ and g in $\mathfrak{C}$ such that $s_2 g = s_1 t$:



We set $s := s_2 g = s_1 t$, $f_1 := a_1 t$ and $f_2 := a_2 g$. Hence $a_1 s_1^{-1} = f_1 s^{-1}$ and $a_2 s_2^{-1} = f_2 s^{-1}$.

We define

$$a_1 s_1^{-1} + a_2 s_2^{-1} := (f_1 + f_2) s^{-1}.$$

One can prove that this sum is well defined and that it makes composition bilinear.

The triangulated structure on the localization

Definition 2.7.

Let $\mathbb{C}$ be a triangulated category. A multiplicative system S in $\mathfrak{C}$ is said to be **compatible with the triangulation** if the following two axioms are satisfied:

MST4: $s \in S$ if and only if $T(s) \in S$, where T is the translation functor.

MST5: Given two exact triangles (X, Y, Z, u, v, w) and $(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{u}, \tilde{v}, \tilde{w})$ with morphisms $f\colon X \to \tilde{X}$, $g\colon Y \to \tilde{Y}$ in S such that $gu = \tilde{u}f$, then there exists a morphism $h\colon Z \to \tilde{Z}$ in S such that (f, g, h) is a morphism of triangles.

The triangulated structure on the localization

Proposition 2.8.

Let $\mathfrak{C}$ be a triangulated category and let S be a multiplicative system compatible with the triangulation. Then $(\mathfrak{C}_S, Q)$ has a structure of triangulated category such that Q is a triangulated functor.

Let $T_{\mathfrak{C}}$ be the translation functor on $\mathfrak{C}$. We define a translation functor $T_{\mathfrak{C}_S}$ on $\mathfrak{C}_S$ as follows:

- $T_{\mathfrak{C}_S}(A) := T_{\mathfrak{C}}(A)$ for every $A \in \mathrm{Ob}(\mathfrak{C}_S)$;
- $T_{\mathfrak{C}_S}(as^{-1}) := T_{\mathfrak{C}}(a)T_{\mathfrak{C}}(s)^{-1}$ for every morphism as^{-1}.

Hence we have that $QT_{\mathfrak{C}} = T_{\mathfrak{C}_S}Q$.

We then define an exact triangle in $\mathfrak{C}_S$ to be any triangle $(A, B, C, a_1s_1^{-1}, a_2s_2^{-1}, a_3s_3^{-1})$ isomorphic to $(Q(X), Q(Y), Q(Z), Q(u), Q(v), Q(w))$, where (X, Y, Z, u, v, w) is an exact triangle in $\mathfrak{C}$.

One can prove that $\mathfrak{C}_S$ is a triangulated category. Q is then a triangulated functor by construction.

The triangulated structure on the localization

Proposition 2.9.

Let $\mathfrak{C}$ be a category and let S be a multiplicative system in $\mathfrak{C}$. Let $\mathfrak{D}$ be a full subcategory of $\mathfrak{C}$ such that the restriction $S \cap \mathfrak{D}$ is a multiplicative system in $\mathfrak{D}$. Assume furthermore that one of the following two conditions holds:

i) *If $s\colon X' \to X$ is in S with $X \in \mathrm{Ob}(\mathfrak{D})$, then there exists a morphism $f\colon X'' \to X'$ with $X'' \in \mathrm{Ob}(\mathfrak{D})$ such that $sf \in S$.*

ii) *If $t\colon X \to X'$ is in S with $X \in \mathrm{Ob}(\mathfrak{D})$, then there exists a morphism $g\colon X' \to X''$ with $X'' \in \mathrm{Ob}(\mathfrak{D})$ such that $gt \in S$.*

Then the natural functor $\mathfrak{D}_{S\cap\mathfrak{D}} \to \mathfrak{C}_S$ is full and faithful (i.e. $\mathfrak{D}_{S\cap\mathfrak{D}}$ can be identified with a full subcategory of $\mathfrak{C}_S$).

Part III

The derived category

Proposition 3.1.

Let $\mathfrak{C}$ be a triangulated category, $\mathfrak{A}$ an abelian category and $H\colon \mathfrak{C} \to \mathfrak{A}$ a cohomological functor. Let S be the collection of morphisms s in $\mathfrak{C}$ such that $H(T^i(s))$ is an isomorphism for all $i \in \mathbb{Z}$. Then S is a multiplicative system compatible with the triangulation in $\mathfrak{C}$.

We show that all the axioms hold.

MS1:
- For each $X \in \mathrm{Ob}(\mathfrak{C})$ we have that id_X is an isomorphism, hence $H(T^i(\mathrm{id}_X))$ is an isomorphism for all $i \in \mathbb{Z}$; it follows that $\mathrm{id}_X \in S$.
- If $f, g \in S$ and fg exists, then $H(T^i(fg)) = H(T^i(f))H(T^i(g))$ is an isomorphism for every $i \in \mathbb{Z}$, since $H(T^i(f))$ and $H(T^i(g))$ are isomorphisms for every $i \in \mathbb{Z}$.

MST4: $s \in S \iff H(T^i(s))$ is an isomorphism for every $i \in \mathbb{Z}$
$\iff H(T^{i-1}(T(s)))$ is an isomorphism for every $i \in \mathbb{Z}$
$\iff T(s) \in S$.

Quasi-isomorphisms

MST5: Suppose that we have the commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & T(X) \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow T(f) \\ \tilde{X} & \xrightarrow{\tilde{u}} & \tilde{Y} & \xrightarrow{\tilde{v}} & \tilde{Z} & \xrightarrow{\tilde{w}} & T(\tilde{X}) \end{array}$$

with $f, g \in S$ and h is a morphism induced by f and g by TR3. Applying $H(T^i(-))$ we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} H(T^i(X)) & \to & H(T^i(Y)) & \to & H(T^i(Z)) & \to & H(T^{i+1}(X)) & \to & H(T^{i+1}(Y)) \\ {\scriptstyle H(T^i(f))}\downarrow\cong & & {\scriptstyle H(T^i(g))}\downarrow\cong & & {\scriptstyle H(T^i(h))}\downarrow & & {\scriptstyle H(T^{i+1}(f))}\downarrow\cong & & {\scriptstyle H(T^{i+1}(g))}\downarrow\cong \\ H(T^i(\tilde{X})) & \to & H(T^i(\tilde{Y})) & \to & H(T^i(\tilde{Z})) & \to & H(T^{i+1}(\tilde{X})) & \to & H(T^{i+1}(\tilde{Y})) \end{array}$$

Hence by the Five lemma, $H(T^i(h))$ is an isomorphism for every $i \in \mathbb{Z}$.

MS2: If we have the diagram

$$\begin{array}{ccc} & & Z \\ & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array}$$

with $s \in S$, then by TR1 there is an exact triangle

$$Z \xrightarrow{s} Y \xrightarrow{f} N \xrightarrow{g} T(Z)$$

and so by TR2

$$Y \xrightarrow{f} N \xrightarrow{g} T(Z) \xrightarrow{-T(s)} T(Y)$$

is an exact triangle. Again by TR1

$$\begin{array}{ccc} fu : X & \longrightarrow & N \\ {\scriptstyle u}\searrow & & \nearrow{\scriptstyle f} \\ & Y & \end{array}$$

can be embedded in an exact triangle $X \xrightarrow{fu} N \xrightarrow{h} W \xrightarrow{t} T(X)$.

Quasi-isomorphisms

Hence we have a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{fu} & N & \xrightarrow{h} & W & \xrightarrow{t} & T(X) \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \mathrm{id}_N} & & \vdots{\scriptstyle v} & & \downarrow{\scriptstyle T(u)} \\
Y & \xrightarrow{f} & N & \xrightarrow{g} & T(Z) & \xrightarrow{-T(s)} & T(Y)
\end{array}$$

with $v\colon W \to T(Z)$ given by TR3.

It follows that $T(u)t = -T(s)v$ and so $uT^{-1}(t) = -sT^{-1}(v)$. We set $W' := T^{-1}(W)$, $u' := -T^{-1}(v)$ and $s' := T^{-1}(t)$. Thus there is a commutative diagram

$$\begin{array}{ccc}
W' & \xrightarrow{s'} & X \\
\downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\
Z & \xrightarrow{s} & Y.
\end{array}$$

Now we show that $s' = T^{-1}(t) \in S$.

Quasi-isomorphisms

Since $Z \xrightarrow{s} Y \xrightarrow{f} N \xrightarrow{g} T(Z)$ is an exact triangle and H is a cohomological functor, the sequence

$$\cdots \to H(T^i(Z)) \xrightarrow{H(T^i(s))} H(T^i(Y)) \xrightarrow{H(T^i(f))} H(T^i(N)) \xrightarrow{H(T^i(g))} H(T^{i+1}(Z)) \xrightarrow{H(T^{i+1}(s))} \cdots$$

is exact. Furthermore, since $s \in S$, $H(T^i(s))$ is an isomorphism for every $i \in \mathbb{Z}$, and so $H(T^i(N)) = 0$ for every $i \in \mathbb{Z}$.

Using the exact triangle $X \xrightarrow{fu} N \xrightarrow{h} W \xrightarrow{t} T(X)$ we have another long exact sequence:

$$\cdots \longrightarrow 0 \xrightarrow{H(T^i(h))} H(T^i(W)) \xrightarrow{H(T^i(t))} H(T^{i+1}(X)) \xrightarrow{H(T^{i+1}(fu))} 0 \longrightarrow \cdots$$

Hence $H(T^i(t))$ is an isomorphism for every $i \in \mathbb{Z}$ and $t \in S$. By MST4, $s' \in S$.

The other part of the axiom is analogous.

MS3: Let $g_1, g_2 \colon X \to Y$ be two morphisms in $\mathfrak{C}$. We consider the morphism $f = g_1 - g_2 \colon X \to Y$ and we show that the following two conditions are equivalent:

i) there exists $s \colon Y \to Y'$ with $s \in S$ such that $sf = 0$;
ii) there exists $t \colon X' \to X$ with $t \in S$ such that $ft = 0$.

Suppose that $i)$ is satisfied. By TR1 and TR2, $s \colon Y \to Y'$ can be embedded in an exact triangle

$$Z \xrightarrow{v} Y \xrightarrow{s} Y' \longrightarrow T(Z).$$

Using the cohomological functor Hom $(X, -)$ we obtain the (long) exact sequence

$$\mathrm{Hom}\,(X, Z) \xrightarrow{v \circ -} \mathrm{Hom}\,(X, Y) \xrightarrow{s \circ -} \mathrm{Hom}\,(X, Y').$$

Since $f \in \mathrm{Hom}\,(X, Y)$ and $sf = 0$, we have that $f \in \ker(s \circ -) = \mathrm{Im}(v \circ -)$ and so there exists $g \colon X \to Z$ such that $f = vg$.

Again by TR1 and TR2, g can be embedded in an exact triangle

$$X' \xrightarrow{t} X \xrightarrow{g} Z \longrightarrow T(X').$$

And again using Hom $(-, Y)$ we obtain the exact sequence

$$\text{Hom }(Z, Y) \xrightarrow{-\circ g} \text{Hom }(X, Y) \xrightarrow{-\circ t} \text{Hom }(X', Y).$$

Since $v \in$ Hom (Z, Y), we have
$f = vg \in \text{Im}(-\circ g) = \ker(-\circ t)$ and so $ft = 0$.
Now we show that $t \in S$. Since $Z \xrightarrow{v} Y \xrightarrow{s} Y' \to T(Z)$ is an exact triangle, we have a long exact sequence

$$H(T^{i-1}(Y)) \xrightarrow{H(T^{i-1}(s))} H(T^{i-1}(Y')) \longrightarrow H(T^i(Z)) \longrightarrow H(T^i(Y)) \xrightarrow{H(T^i(s))} H(T^i(Y')).$$

Since $s \in S$, $H(T^i(Z)) = 0$ for every $i \in \mathbb{Z}$.

Quasi-isomorphisms

Using the exact triangle

$$X' \xrightarrow{t} X \xrightarrow{g} Z \longrightarrow T(X').$$

we have that

$$0 \longrightarrow H(T^i(X')) \xrightarrow{H(T^i(t))} H(T^i(X)) \longrightarrow 0$$

is exact. It follows that $t \in S$.

The implication $ii) \Rightarrow i)$ is analogous.

The derived category

Corollary 3.2.

Let $\mathfrak{A}$ be an abelian category and $H\colon \mathcal{K}(\mathfrak{A}) \to \mathfrak{A}$ the cohomological functor defined in Proposition 1.15. Let Qis *be the collection of all* **quasi-isomorphisms** *s, i.e. such that $H(T^i(s))$ is an isomorphism for every $i \in \mathbb{Z}$, in $\mathcal{K}(\mathfrak{A})$.*
Then Qis *is a multiplicative system compatible with the triangulation in $\mathcal{K}(\mathfrak{A})$.*

Similarly, $\mathrm{Qis} \cap \mathcal{K}^+(\mathfrak{A})$ (respectively $\mathrm{Qis} \cap \mathcal{K}^-(\mathfrak{A})$, $\mathrm{Qis} \cap \mathcal{K}^{\mathrm{b}}(\mathfrak{A})$) is a multiplicative system compatible with the triangulation in $\mathcal{K}^+(\mathfrak{A})$ (respectively $\mathcal{K}^-(\mathfrak{A})$, $\mathcal{K}^{\mathrm{b}}(\mathfrak{A})$).

The first assertion is immediate from Proposition 3.1. The last one follows from Proposition 2.9.

Definition 3.3.

Let $\mathfrak{A}$ be an abelian category. We define the **derived category** of $\mathfrak{A}$, denoted $\mathcal{D}(\mathfrak{A})$, as

$$\mathcal{D}(\mathfrak{A}) := \mathcal{K}(\mathfrak{A})_{\mathsf{Qis}}$$

Similarly, we define $\mathcal{D}^{+}(\mathfrak{A}) := \mathcal{K}^{+}(\mathfrak{A})_{\mathsf{Qis}\cap\mathcal{K}^{+}(\mathfrak{A})}$,
$\mathcal{D}^{-}(\mathfrak{A}) := \mathcal{K}^{-}(\mathfrak{A})_{\mathsf{Qis}\cap\mathcal{K}^{-}(\mathfrak{A})}$ and $\mathcal{D}^{\mathsf{b}}(\mathfrak{A}) := \mathcal{K}^{\mathsf{b}}(\mathfrak{A})_{\mathsf{Qis}\cap\mathcal{K}^{\mathsf{b}}(\mathfrak{A})}$.

Remark 3.4.

Let $\mathfrak{A}$ be an abelian category. By Proposition 2.9, $\mathcal{D}^{+}(\mathfrak{A})$, $\mathcal{D}^{-}(\mathfrak{A})$ and $\mathcal{D}^{\mathsf{b}}(\mathfrak{A})$ are full subcategories of $\mathcal{D}(\mathfrak{A})$.
By Proposition 2.8, $\mathcal{D}(\mathfrak{A})$, $\mathcal{D}^{+}(\mathfrak{A})$, $\mathcal{D}^{-}(\mathfrak{A})$ and $\mathcal{D}^{\mathsf{b}}(\mathfrak{A})$ are all triangulated categories.

Part IV

Verdier quotients

Split triangles

Remark 4.1.

Let $(\mathcal{T}, T)$ be a triangulated category. Given two exact triangles of the trivial form $X \xrightarrow{\text{id}} X \to 0 \to T(X)$ and $Z \xrightarrow{\text{id}} Z \to 0 \to T(Z)$, we can construct a new exact triangle summing them, but this is just the trivial triangle on the sum object.

Instead we can first rotate one of the two and then sum them, to obtain a triangle of the form

$$X \xrightarrow{(\text{id},0)} X \oplus Z \xrightarrow{0+\text{id}} Z \xrightarrow{0} T(X)$$

with the canonical morphisms, which one can show is an exact triangle. We say that a triangle **splits** if it is isomorphic to a triangle of this kind.

Split triangles

Proposition 4.2.

Let $\mathcal{T}$ be a triangulated category. Any exact triangle with a zero map splits.

Let $(X, Y, Z, u, v, 0)$ be an exact triangle. As we saw in Remark 4.1, there is an exact triangle of the form

$$X \xrightarrow{(\mathrm{id},0)} X \oplus Z \xrightarrow{0+\mathrm{id}} Z \xrightarrow{0} T(X)$$

and a commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & X \oplus Z & \longrightarrow & Z & \xrightarrow{0} & T(X) \\
 & & & & \big\downarrow \mathrm{id} & & \big\downarrow T(\mathrm{id}) \\
X & \longrightarrow & Y & \longrightarrow & Z & \xrightarrow{0} & T(X)
\end{array}$$

Using TR2 and TR3 we can complete it to a morphism of triangles. By Proposition 1.8 we obtain that the two triangles are isomorphic.

Thick subcategories

Definition 4.3.

Let $\mathcal{T}$ be a triangulated category. A triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is said **thick** if it contains all the direct summands of its objects.

Example 4.4.

Let $\mathfrak{A}$ be an abelian category and consider the triangulated category $\mathcal{K}(\mathfrak{A})$. The full subcategory $\mathcal{E}$ of $\mathcal{K}(\mathfrak{A})$ of exact complexes is a thick triangulated subcategory.
(It is thick because homology commutes with direct sums)

Definition 4.5.

Let $F\colon \mathfrak{D} \to \mathcal{T}$ be a triangulated functor. We define the **kernel** of F, denoted $\mathrm{Ker}(F)$, to be the full subcategory $\mathcal{K}$ of $\mathfrak{D}$ of objects mapped to the zero object.

Proposition 4.6.

The kernel $\mathcal{K}$ of a triangulated functor $F\colon \mathfrak{D} \to \mathcal{T}$ is a thick subcategory of $\mathfrak{D}$.

Since the translation functor T commutes with F, we that the property for an object to be in the kernel of F is respected by T and obviously the kernel is closed under isomorphism. Now let us see that given an exact triangle $X \to Y \to Z \to T(X)$, in $\mathfrak{D}$, with X and Y in the kernel, Z has to be in the kernel too.

Since F is triangulated, $F(Z)$ fits in an exact triangle of the kind

$$0 \to 0 \to F(Z) \to T(0)$$

and therefore is 0 by Proposition 4.2. Finally, a direct summand of an object in the kernel is in the kernel because of the additivity of F.

Remark 4.7.

We have just seen that kernels are always thick subcategories. What about the converse?
This question brings to define Verdier quotients.

Lemma 4.8.

Let $\mathfrak{D}$ be a triangulated category and let $\mathfrak{C}$ be a full triangulated subcategory. Set

$$S_{\mathfrak{C}}^{\mathfrak{D}} := \{f \in \mathrm{Arrows}(\mathfrak{D}) \text{ such that there exists an exact triangle } (X, Y, Z, f, v, w) \text{ of } \mathfrak{D} \text{ with } Z \text{ isomorphic to an object of } \mathfrak{C}\}$$

Then $S_{\mathfrak{C}}^{\mathfrak{D}}$ is a multiplicative system compatible with the triangulation in $\mathfrak{D}$.

Let's prove all the axioms.

MS1: If $X \in \mathrm{Ob}(\mathfrak{D})$ then $(X, X, 0, \mathrm{id}, 0, 0)$ is an exact triangle and 0 is an object of $\mathfrak{C}$; thus $\mathrm{id}_X \in S_{\mathfrak{C}}^{\mathfrak{D}}$.
Let now $f\colon X \to Y$ and $g\colon Y \to Z$ be composable morphisms contained in $S_{\mathfrak{C}}^{\mathfrak{D}}$. Choose exact triangles (X, Y, Q_1, f, p_1, d_1), $(X, Z, Q_2, g \circ f, p_2, d_2)$ and (Y, Z, Q_3, g, p_3, d_3).

Verdier quotient

By assumption we know that Q_1 and Q_3 are isomorphic to objects of $\mathfrak{C}$. By TR4 we know that there exists an exact triangle (Q_1, Q_2, Q_3, a, b, c). Since $\mathfrak{C}$ is a triangulated subcategory we conclude that Q_2 is isomorphic to an object of $\mathfrak{C}$. Hence $g \circ f \in S_{\mathfrak{C}}^{\mathfrak{D}}$.

MS3: We prove just one implication, the other one is dual. Let $a \colon X \to Y$ be a morphism and let $t \colon Z \to X$ be an element of $S_{\mathfrak{C}}^{\mathfrak{D}}$ such that $a \circ t = 0$. It suffices to find an $s \in S_{\mathfrak{C}}^{\mathfrak{D}}$ such that $s \circ a = 0$. Choose an exact triangle (Z, X, Q, t, g, h) using TR1 and TR2. Since $a \circ t = 0$ we see by Example 1.7 that there exists $i \colon Q \to Y$ such that $i \circ g = a$.

Verdier quotient

Finally, using TR1 again we choose an exact triangle (Q, Y, W, i, s, k):

$$
\begin{array}{ccccccc}
Z & \xrightarrow{t} & X & \xrightarrow{g} & Q & \longrightarrow & T(Z) \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle i} & & \\
 & & X & \xrightarrow{a} & Y & & \\
 & & & & \downarrow{\scriptstyle s} & & \\
 & & & & W. & &
\end{array}
$$

Since $t \in S_{\mathfrak{C}}^{\mathfrak{D}}$, we see that Q is isomorphic to an object of $\mathfrak{C}$. Hence $s \in S_{\mathfrak{C}}^{\mathfrak{D}}$. Finally $s \circ a = s \circ i \circ g = 0$ as $s \circ i = 0$ by Remark 1.4.

MST4: Follows from the fact that exact triangles and $\mathfrak{C}$ are stable under translations.

MST5: Suppose given a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} \\ X' & \longrightarrow & Y' \end{array}$$

with $s, s' \in S_{\mathfrak{C}}^{\mathfrak{D}}$. It is easy but tedious to prove that we can extend this commutative square to a nine square diagram

Verdier quotient

$$
\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} & & \downarrow & & \downarrow \\
X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & T(X'') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
T(X) & \longrightarrow & T(Y) & \longrightarrow & T(Z) & \longrightarrow & T(T(X))
\end{array}
$$

where all the squares are commutative, except for the lower right square which is anticommutative, all rows and columns are exact triangles and the morphisms on the bottom row (respectively right column) are the translated of the ones in the first row (left column).

As $s, s' \in S_{\mathfrak{C}}^{\mathfrak{D}}$ we see that X'' and Y'' are isomorphic to objects of $\mathfrak{C}$. Since $\mathfrak{C}$ is a triangulated subcategory we have that Z'' is also isomorphic to an object of $\mathfrak{C}$. Hence the morphism $Z \to Z'$ is an element of $S_{\mathfrak{C}}^{\mathfrak{D}}$.

MS2: This axiom is actually a formal consequence of MS1, MST4 and MST5:
Let $f : X \to Y$ be a morphism of $\mathfrak{D}$ and let $t : X \to X'$ be an element of $S_{\mathfrak{C}}^{\mathfrak{D}}$. Choose exact triangles (X, Y, Z, f, g, h) and $(X', Y', Z, f', g', T(t) \circ h)$, by TR1 and TR2.

Verdier quotient

By MST4 and MST5 (and TR2) we can find the dotted arrow in the commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\
\big\downarrow t & & \big\downarrow s' & & \big\downarrow \mathrm{id} & & \big\downarrow T(t) \\
X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & T(X')
\end{array}$$

with moreover $s' \in S$. This proves one of the points of the axiom. The proof for the other one is dual.

Verdier quotient

Definition 4.9.

Let $\mathfrak{D}$ be a full triangulated category and let $\mathfrak{C}$ be a triangulated subcategory of $\mathfrak{D}$. The **Verdier quotient** of $\mathfrak{D}$ with respect to $\mathfrak{C}$, denoted $\mathfrak{D}/\mathfrak{C}$, is defined to be the localization category $\mathfrak{D}_{S_{\mathfrak{C}}^{\mathfrak{D}}}$ where $S_{\mathfrak{C}}^{\mathfrak{D}}$ is the multiplicative system defined in Lemma 4.8. Moreover the localization functor $\mathfrak{D} \to \mathfrak{D}_{S_{\mathfrak{C}}^{\mathfrak{D}}}$ is called **quotient functor** or **Verdier localization map**.

Verdier quotient

Lemma 4.10.

Let $\mathfrak{D}$ be a triangulated category and let $\mathfrak{C}$ be a full triangulated subcategory of $\mathfrak{D}$. The kernel of the quotient functor $Q\colon \mathfrak{D} \to \mathfrak{D}/\mathfrak{C}$ is the full subcategory of $\mathfrak{D}$ whose objects are

$$\begin{aligned}\mathrm{Ob}(\mathrm{Ker}(Q)) = \{Z \in \mathrm{Ob}(\mathfrak{D}) \text{ such that there exists a } Z' \in \mathrm{Ob}(\mathfrak{D}) \\ \text{such that } Z \oplus Z' \text{ is isomorphic to an object of } \mathfrak{C}\}\end{aligned}$$

In other words, $\mathrm{Ker}(Q)$ *is the smallest full thick subcategory of $\mathfrak{D}$ containing $\mathfrak{C}$.*

From Proposition 4.6, we know that $\mathrm{Ker}(F)$ is a full triangulated subcategory containing summands of any of its objects. To conclude, it then suffices to show that the following three statements are equivalent for any S multiplicative system compatible with the triangulation in $\mathfrak{D}$:

(1) $Q(Z) = 0$ in $\mathfrak{D}_S$;

(2) there exists $Z' \in \mathrm{Ob}(\mathfrak{D})$ such that $0\colon Z \to Z'$ is an element of S;

(3) there exists an object Z' and an exact triangle $(X, Y, Z \oplus Z', f, g, h)$ such that $f \in S$.

"(2)⇒(1)": If (2) holds, $0 = Q(0)\colon Q(Z) \to Q(Z')$ is an isomorphism. Since $\mathfrak{D}_S$ is additive, $Q(Z) = 0$.
"(1)⇒(2)": Assume that $Q(Z) = 0$. This implies that the morphism $f\colon Z \to 0$ is transformed into an isomorphism in $\mathfrak{D}_S$. Then one can show that there exists a morphism $g\colon 0 \to Z'$ such that $fg \in S$.

Verdier quotient

"(2)$\Rightarrow$(3)": If (2) holds, then $(T^{-1}(Z'), T^{-1}(Z') \oplus Z, Z, (\mathrm{id}, 0), 0 + \mathrm{id}, 0)$ is an exact triangle, by Remark 4.1 with $0 \in S$. By rotating we conclude that (3) holds.
"(3)$\Rightarrow$(2)": If $(X, Y, Z \oplus Z', f, g, h)$ is an exact triangle with $f \in S$ then $Q(f)$ is an isomorphism, whence $Q(Z \oplus Z') = 0$ and then $Q(Z) = 0$.

Remark 4.11.

Let $\mathfrak{D}$ be a triangulated category and $\mathfrak{C}$ a thick full triangulated subcategory. Call $Q\colon \mathfrak{D} \longrightarrow \mathfrak{D}/\mathfrak{C} = \mathfrak{D}_{S_{\mathfrak{C}}^{\mathfrak{D}}}$ the quotient functor. From Lemma 4.10 we have that $\mathrm{Ker}(Q)$ is the smallest full thick subcategory of $\mathfrak{D}$ which contains $\mathfrak{C}$. But if $\mathfrak{C}$ is thick, then $\mathrm{Ker}(Q) = \mathfrak{C}$ and thus the Verdier quotient $\mathfrak{D}/\mathfrak{C}$ is actually a quotient.

Verdier quotient

Remark 4.12.

From the properties of the localization, we obtain the universal property of the quotient for the Verdier quotient and also that the quotient of a triangulated category for a thick triangulated subcategory is still a triangulated category.

Part V

Stabilized categories of a ring

The main reference

We will describe now an application of the derived categories, given by Buchweitz in the article "Maximal Cohen-Macaulay Modules and Tate Cohomology over Gorenstein Rings". We will see that maximal Cohen-Macaulay modules, at least up to projective modules, carry a natural triangulated structure.

Notations and Conventions 5.1.

This theory does not require the commutativity of the ring. All rings considered will be associative with unit, left and right noetherian; all modules will be unitary.
We will denote $\mathcal{M}od\text{-}S$ the category of all right S-modules and module homomorphisms, and mod-S the full subcategory of $\mathcal{M}od\text{-}S$ of finitely generated S-modules.
We will then denote $\mathcal{P}(S)$ the full subcategory of $\mathcal{M}od\text{-}S$ of finitely generated projective modules, and $\mathcal{D}^*(S)$ for $*$ in $\{\ \ , +, -, \mathrm{b}\}$ the derived categories of $\mathcal{M}od\text{-}S$ whose objects are all complexes of S-modules with finitely generated cohomology modules. Correspondingly, $\mathcal{K}^*(S)$ will denote the homotopy categories of those complexes of S-modules whose cohomology modules are finitely generated.
Each $\mathcal{D}^*(S)$ and $\mathcal{K}^*(S)$ will be considered a triangulated category with respect to its natural triangulated structure.
We will mostly focus on $\mathcal{D}^{\mathrm{b}}(S)$ which is, by the conventions described above, the derived category of all complexes of S-modules with finitely generated total cohomology.

Perfect complexes

Definition 5.2.

A complex of S-modules is said **perfect** if it is isomorphic in $\mathcal{D}(S)$ to a finite complex of finitely generated projective S-modules.
Perfect complexes form an essential (i.e. closed under isomorphisms), full and triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(S)$, denoted $\mathcal{D}^{\mathrm{b}}_{\mathrm{perf}}(S)$

Remark 5.3.

An S-module is perfect (considered as a complex concentrated on degree zero) if and only if its projective dimension is finite.

Perfect complexes

Lemma 5.4.

Let X be an object in $\mathcal{D}^{\mathrm{b}}(S)$. Then the following conditions are equivalent:

(i) *X is perfect;*

(ii) *There is an integer $\mathrm{i}(X)$ such that for any $i \geq \mathrm{i}(X)$ and any finitely generated S-module M we have that*

$$\mathrm{Ext}^i_S(X, M) := \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(S)}(X, T^i(M)) = 0$$

(iii) *For any triangulated functor $F\colon \mathcal{D}^{\mathrm{b}}(S) \to \mathcal{D}$ into another triangulated category $\mathcal{D}$ for which $F(S) = 0$, where S is considered a (complex of) right module(s) over itself, one has that $F(X) = 0$.*

See [Buchweitz, "Maximal Cohen-Macaulay Modules and Tate-Cohomology over Gorenstein Rings", Lemma 1.2.1]

Stabilized derived category

Corollary 5.5.

$\mathcal{D}^{\mathrm{b}}_{\mathrm{perf}}(S)$ is a thick triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(S)$.

Let X be in $\mathcal{D}^{\mathrm{b}}_{\mathrm{perf}}(S)$ and let Y be a direct summand of X. Then, since $\mathrm{Ext}^i_S(-, M) = \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(S)}(-, T^i(M))$ is an additive functor, we obtain that $\mathrm{Ext}^i_S(Y, M)$ is a direct summand of $\mathrm{Ext}^i_S(X, M)$. Now the equivalence between (1) and (2) of Lemma 5.4 implies that Y is in $\mathcal{D}^{\mathrm{b}}_{\mathrm{perf}}(S)$.

Remark 5.6.

From Corollary 5.5 and Remarks 4.11 and 4.12 we can then consider the Verdier quotient $\mathcal{D}^{\mathrm{b}}(S)/\mathcal{D}^{\mathrm{b}}_{\mathrm{perf}}(S)$ and this will satisfy the universal property of the quotient and will be a triangulated category.

Definition 5.7.

The triangulated quotient category

$$\underline{\mathcal{D}^{\mathrm{b}}(S)} = \mathcal{D}^{\mathrm{b}}(S)/\mathcal{D}^{\mathrm{b}}_{\mathrm{perf}}(S)$$

will be called the **stabilized derived category** or **singularity category** of S.

Definition 5.8.

The **projectively stabilized category** of finitely generated S-modules, denoted $\underline{\mathrm{mod}\text{-}S}$, is defined by factoring out all projective modules from $\mathrm{mod}\text{-}S$, i.e. $\underline{\mathrm{mod}\text{-}S}$ has the same objects as $\mathrm{mod}\text{-}S$ and its morphisms are given by

$$\underline{\mathrm{Hom}}_S(M,N) := \mathrm{Hom}_{\underline{\mathrm{mod}\text{-}S}}(M,N) = \frac{\mathrm{Hom}_S(M,N)}{\{f\colon M \to N \mid \text{f factors through a projective } S\text{-module}\}}$$

for all finitely generated S-modules M and N.

Stably equivalent modules

Remark 5.9.

$\underline{\text{mod-}S}$ is still an additive category and two S-modules M and N are isomorphic in $\underline{\text{mod-}S}$ if and only if they are **stably equivalent** (by projectives), i.e. $\exists\, P, Q$ finitely generated projective S-modules such that $M \oplus P \cong_S N \oplus Q$.

"$\Rightarrow$": Assume M and N are isomorphic in $\underline{\text{mod-}S}$. Then there exist $f\colon M \to N$ and $g\colon N \to M$ such that $gf - \mathrm{id}_M\colon M \xrightarrow{\phi_1} P \xrightarrow{\phi_2} M$ with P projective and $fg - \mathrm{id}_N\colon N \xrightarrow{\psi_1} Q \xrightarrow{\psi_2} N$ with Q projective.
Now we construct $M \xrightarrow{f'} N \oplus P \xrightarrow{g'} M$ such that $g'f' = \mathrm{id}_M$ and $f'g' - \mathrm{id}_{N\oplus P}$ factors through a projective.
For this, we define $f'(m) := (f(m), \phi_1(m))$ and $g'(n, p) := g(n) - \phi_2(p)$.
Then $f'g' - \mathrm{id}_{N\oplus P}\colon N \oplus P \xrightarrow{\theta_1} Q \oplus P \oplus P \xrightarrow{\theta_2} N \oplus P$ with $Q \oplus P \oplus P$ projective, $\theta_1(n, p) := (\psi_1(n), p, \phi_1 \circ g(n) - \phi_1(\phi_2(p)))$ and $\theta_2(q, p_1, p_2) := (\psi_2(q) - f \circ \phi_2(p_1), p_2 - p_1)$.

Stably equivalent modules

Now we have that $M \oplus \mathrm{Ker}(g') \cong N \oplus P$, and then, using the fact that $f'g' - \mathrm{id}_{N\oplus P}$ factors through a projective, that $\mathrm{Ker}(g')$ is projective.
In fact $\mathrm{Ker}(g')$ is a direct summand of $N \oplus P$, and writing i and π for the inclusion and projection maps, we have that

$$\pi \circ (\mathrm{id}_{N\oplus P} - f'g') \circ i = \mathrm{id}_{\mathrm{Ker}(g')}.$$

Since $\mathrm{id}_{N\oplus P} - f'g'$ factors through a projective, so does $\mathrm{id}_{\mathrm{Ker}(g')}$ and so $\mathrm{Ker}(g')$ is a direct summand of a projective and therefore projective.
"$\Leftarrow$": Assume now that $M \oplus P \cong_S N \oplus Q$ with P and Q projective and call φ the isomorphism. Now consider the maps

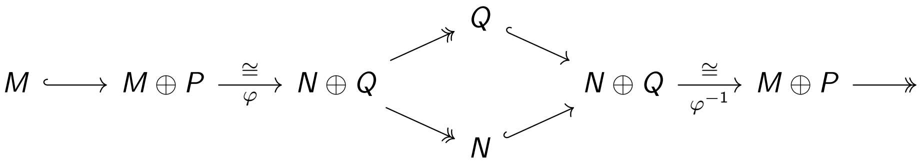
$$M \hookrightarrow M \oplus P \xrightarrow[\varphi]{\cong} N \oplus Q \twoheadrightarrow N \hookrightarrow N \oplus Q \xrightarrow[\varphi^{-1}]{\cong} M \oplus P \twoheadrightarrow M$$

where the composite $M \to N$ is f and the composite $N \to M$ is g.

We are done if we prove that $gf - \mathrm{id}_M$ and $fg - \mathrm{id}\, N$ factor through a projective. But, for simmetry, it suffices to show that just $gf - \mathrm{id}_M$ factors through a projective.

Stably equivalent modules

Now consider the map in the upper part of the diagram, which we call $q\colon M \to M$:



while the map in the lower part of the diagram is $gf\colon M \to M$.
It is easy to see that $gf + q = \mathrm{id}_M$. Then $gf - \mathrm{id}_M = -q$ and $-q$ is an S-linear map which factors through a projective, namely Q.

Definition 5.10.

The **loop-space functor** Ω_S on $\underline{\text{mod-}S}$ is defined by

$$\Omega_S M = \text{Ker}(p_M)$$

for every $M \in \text{mod-}S$, where p_M is a chosen surjection $p_M \colon P_M \to M$ with P_M a finitely generated projective S-module.

Lemma 5.11.

The composition

$$\text{mod-}S \to \mathcal{D}^{\text{b}}(S) \to \underline{\mathcal{D}^{\text{b}}(S)}$$

factors uniquely through the canonical quotient functor $\text{mod-}S \to \underline{\text{mod-}S}$ *and hence yields a naturally defined functor*

$$\iota_S \colon \underline{\text{mod-}S} \to \underline{\mathcal{D}^{\text{b}}(S)}.$$

It transforms the loop-space functor Ω_S *into the inverse of the translation functor on* $\underline{\mathcal{D}^{\text{b}}(S)}$*.*

Complete resolutions

Definition 5.12.

Let $M \in \mathcal{M}od\text{-}S$. Then a **complete resolution** of M (over S) is an acyclic complex (A, d_A) of finitely generated projective S-modules such that

$$\mathrm{Cok}(d_A^0 \colon A^{-1} \to A^0) = M.$$

To abbreviate notations, the complex $A_- = (A^{\leq 0}, \mathrm{d}_A\,|_{A^{\leq 0}})$ with its natural induced augmentation onto M is called the **associated projective resolution** of M, whereas the complex $A_+ = (A^{\geq 1}, \mathrm{d}_A\,|_{A^{\geq 1}})[1]$ with its natural induced augmentation onto M into it is the **associated projective co-resolution** of M.

The category of complete resolutions

Remark 5.13.

M is necessarily finitely generated if it admits a complete resolution, and a finitely generated module admits a complete resolution if and only if it admits a projective co-resolution.
The complete resolution A itself is obtained as the translated mapping cone of $-\mathrm{d}^0 \colon A_- \to A_+$, i.e. $A = \mathrm{C}(-\mathrm{d}^0)[1]$, so that $-\mathrm{d}^0$ serves as the connecting homomorphism from the associated projective resolution to the associated projective co-resolution.

Definition 5.14.

The **category of complete resolutions**, denoted $\underline{\mathrm{APC}(S)}$, is defined as the homotopy category of acyclic complexes of finitely generated projective S-modules.

Remark 5.15.

$\underline{\mathrm{APC}(S)}$ is a full triangulated subcategory of $\mathcal{K}(S)$, so it inherits a triangulated structure from $\mathcal{K}(S)$.

Definition 5.16.

We define now a collection of functors on $\underline{\mathrm{APC}(S)}$:
for any complex (X, id_X) in $\mathcal{K}(S)$ set

$$\Omega_i(X) = \mathrm{Cok}(\mathrm{d}_X^{-i} \colon X^{-i-1} \to X^{-i}) \quad \text{for any } i \in \mathbb{Z}.$$

We will call $\Omega_i(X)$ defined above the ***i*-th syzygy module** of X.

Remark 5.17.

As obviously

$$\Omega_0(X[1]) = \mathrm{Cok}(-\,\mathrm{d}_X^{-1} \colon X^0 \to X^1) \simeq \Omega_{-1}(X)$$

one obtains

$$\Omega_i(X[j]) \simeq \Omega_{i-j}(X)$$

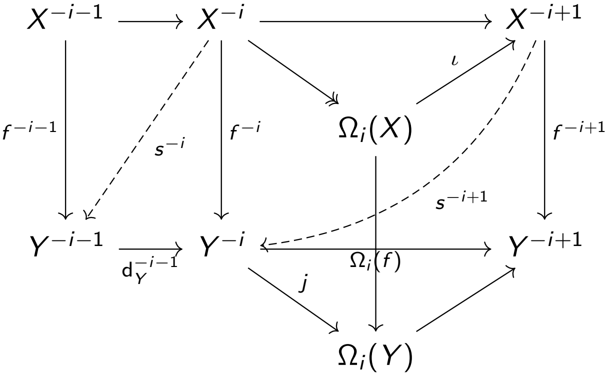
for all integers i, j and all complexes X.
Anyway this isomorphism of functors is not canonical.

Lemma 5.18.

Each Ω_i defines a functor from $\underline{\mathrm{APC}(S)}$ into $\underline{\mathrm{mod}\text{-}S}$. It transforms the inverse of the translation functor on $\underline{\mathrm{APC}(S)}$ into the loop-space functor Ω_S on $\underline{\mathrm{mod}\text{-}S}$.

The category of complete resolutions

If $X \to Y$ is a morphism of complexes of finitely generated projective S-modules which is zero-homotopic, then, for any i, the induced morphism of S-modules $\Omega_i(f)\colon \Omega_i(X) \to \Omega_i(Y)$ factors over a finitely generated projective S-module - namely over both X^{-i+1} and Y^{-i}. In fact we have the following diagram:



The category of complete resolutions

where the dashed arrows are the morphisms which represent the homotopy equivalence with 0, and our claim follows using that all the squares are commutative, that the two triangles with Ω_i commute and that $j \circ \mathrm{d}_Y^{-i-1} \circ s^{-i} = 0$.

The naive filtration functors

Construction 5.19.

Now we will set up functors from $\underline{\mathrm{APC}(S)}$ to $\underline{\mathcal{D}^{\mathrm{b}}(S)}$. For this, we consider the **naive filtration** associated to a comples X in $\mathcal{K}(S)$: $(\sigma_{\leq k}X)_{k\in\mathbb{Z}}$ given by

$$\begin{array}{ccccccccccccc}
(\sigma_{\leq k+1}X) = X^{\leq k+1} \equiv (\cdots & \to & X^i & \to & \cdots & \to & X^k & \to & X^{k+1} & \to & 0 & \to & \cdots) \\
\downarrow & & \downarrow \mathrm{id}_{X^i} & & & & \downarrow \mathrm{id}_{X^k} & & \downarrow & & \downarrow & & \\
(\sigma_{\leq k}X) = X^{\leq k} \equiv (\cdots & \longrightarrow & X^i & \to & \cdots & \to & X^k & \longrightarrow & 0 & \longrightarrow & 0 & \to & \cdots)
\end{array}$$

We obviously have an equality of functors

$$\sigma_{\leq k} \circ T^i = T^i \circ \sigma_{\leq k+1} \quad \text{for all } k \text{ and } i.$$

The naive filtration functors

Coming back to acyclic complexes of finitely generated projective S-modules, the following lemma is easily established.

Lemma 5.20.

Let A and B be objects of $\underline{\mathrm{APC}(S)}$. Then one has:

- (i) *In $\mathcal{D}(S)$ (the class of) the canonical morphism of complexes $\sigma_{\leq k}A \to (\Omega_{-k}A)[-k]$ becomes an isomorphism, or - equivalently -*
- (i′) *$(\sigma_{\leq k}A)[k]$ is a projective resolution of $\Omega_{-k}A$.*
- (ii) *For any two integers k, l with $k \leq l$, the mapping cone over the natural morphism $\sigma_{\leq l}A \to \sigma_{\leq k}A$ is perfect.*
- (iii) *If $f: A \to B$ is a morphism of complexes which is homotopic to zero, all the induced morphisms $\underline{\sigma_{\leq k}f}: \underline{\sigma_{\leq k}A} \to \underline{\sigma_{\leq k}B}$ in $\underline{\mathcal{D}^{\mathrm{b}}(S)}$ are zero.*

To prove (i), it suffices to say that the morphism of complexes

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & A^i & \longrightarrow & \cdots & \longrightarrow & A^{k-1} & \longrightarrow & A^k & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Cok}(A^{k-1} \to A^k) & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

induces an isomorphism on cohomologies. But it easily follows from the definition of cohomology.
(ii) is not difficult to prove, as we can reduce to have 0 in all places in which $\sigma_{\leq l}A$ and $\sigma_{\leq k}A$ differ, and there are only finitely many places left.

The naive filtration functors

We now prove (*iii*), which is the most difficult part. Consider the commutative diagram of morphisms of complexes

$$\begin{array}{ccc} \sigma_{\leq k} A & \longrightarrow & (\Omega_{-k} A)[-k] \\ \big\downarrow{\scriptstyle \sigma_{\leq k} f} & & \big\downarrow{\scriptstyle (\Omega_{-k} f)[-k]} \\ \sigma_{\leq k} B & \longrightarrow & (\Omega_{-k} B)[-k] \end{array}$$

whose horizontal arrows become isomorphisms in $\mathcal{D}^{\mathrm{b}}(S)$, by (*i*), and hence in $\underline{\mathcal{D}^{\mathrm{b}}(S)}$. But, as observed above, f zero-homotopic implies that $\Omega_{-k} f$ factors over a projective module, and so the class of $(\Omega_{-k} f)[-k]$ is zero in $\underline{\mathcal{D}^{\mathrm{b}}(S)}$.

The naive filtration functors

Corollary 5.21.

The naive truncations $(\underline{\sigma}_{\leq k})_{k\in\mathbb{Z}}$ *define a directed system of functors*

$$\cdots \xrightarrow{\simeq} \underline{\sigma}_{\leq k} \xrightarrow{\simeq} \underline{\sigma}_{\leq k-1} \xrightarrow{\simeq} \cdots$$

from $\underline{\mathrm{APC}(S)}$ *to* $\underline{\mathcal{D}^{\mathrm{b}}(S)}$*, whose transition morphisms are all isomorphisms. In particular, its inverse limit* $\underline{\sigma}_{\leq} = \varprojlim_k \underline{\sigma}_{\leq k}$ *exists and it is a triangulated functor of triangulated categories*

$$\underline{\sigma}_{\leq} \colon \underline{\mathrm{APC}(S)} \to \underline{\mathcal{D}^{\mathrm{b}}(S)}.$$

Part VI

Maximal Cohen-Macaulay modules

Strongly Gorenstein rings

From now on, assume that the ring S, which is still noetherian on both sides, that S is of finite injective dimension both as a left or a right module over itself.

Remark 6.1.

If both the left and right injective dimension of S are finite, they are the same and we will call this common value **injective** or **virtual dimension** of S. We will denote it $\operatorname{vdim} S$.

See [A. Zacks, "Injective Dimension of Semi-Primary Rings"].

Definition 6.2.

We call such rings with finite virtual dimension **strongly Gorenstein**.

This definition is motivated from the fact that in the commutative case it is a stronger notion of what is normally called Gorenstein, and which is implicated by Gorenstein and of finite Krull dimension.

Definition 6.3.

Let $\tilde{S}$ be a ring. We call **the opposite ring** of $\tilde{S}$ the ring which has the same elements of S and the same addition, but multiplication performed in the reverse order. More explicitly, the opposite of a ring $(\tilde{S}, +, \cdot)$ is the ring $(\tilde{S}, +, *)$, whose multiplication is defined as $a * b := b \cdot a$ for every $a, b \in \tilde{S}$.
Note that this is exactly what we expect if we consider the category associated to a ring.

Remark 6.4.

A ring S is strongly Gorenstein if and only if this holds for S^{op}, the opposite ring of S.

It follows from the left-right simmetry of the definition of strongly Gorenstein.

Definition 6.5.

Let S be a noetherian ring of finite injective dimension. Then a finitely generated S-module is **maximal Cohen-Macaulay** (MCM for short) if

$$\mathrm{Ext}^i_S(M, S) = 0 \quad \text{for } i \neq 0.$$

The full subcategory of maximal Cohen-Macaulay modules in $\text{mod-}S$ is denoted $\mathrm{MCM}(S)$, and accordingly, its image in $\underline{\text{mod-}S}$ by $\underline{\mathrm{MCM}(S)}$.

To keep the definition "coordinate-free" one may replace the module S in the above definition by any faithfully projective (right) S-module P:

$$M \in \text{mod-}S \text{ is MCM if and only if } \mathrm{Ext}^i_S(M, P) = 0 \text{ for } i \neq 0,$$

where faithfully projective means that $\mathrm{Hom}(P, -)$ is faithful, exact and preserves products.

Maximal Cohen-Macaulay modules

Remark 6.6.

Again the terminology is borrowed from commutative algebra, as over a local, commutative Gorenstein ring it coincides with the usual notion, i.e. Cohen-Macaulay and such that the dimension equals the dimension of the ring, which is the maximal possible.

Example 6.7.

Let $S = k[x,y]/(x^2, xy)$, $M = S/(x)$. Then M is maximal Cohen-Macaulay over S.

We have that $(x^2, xy) = (x^2, x) \cap (x^2, y) = (x) \cap (x^2, y)$. So we immediatly see that $\dim(S) = 1$. Now observe that $M = S/(x) \cong k[x,y]/(x, xy) \cong k[x,y]/(x) \cong k[y]$ and so $\dim(M) = 1 = \operatorname{depth}(M)$. Thus M is maximal Cohen-Macaulay.

Maximal Cohen-Macaulay modules

Lemma 6.8.

Let S be a ring which is strongly Gorenstein. Then

(i) *Any finitely generated projective S-module is maximal Cohen-Macaulay, i.e. $\mathcal{P}(S)$ is a full subcategory of* $\mathrm{MCM}(S)$.

(ii) *If* $0 \to M_1 \xrightarrow{f} M_2 \to M_3 \to 0$ *is an exact sequence in* mod-S, *then*

- M_2, M_3 *in* $\mathrm{MCM}(S)$ *implies that* M_1 *is* MCM,
- M_1, M_3 *in* $\mathrm{MCM}(S)$ *implies that* M_2 *is* MCM.
- M_1, M_2 *in* $\mathrm{MCM}(S)$ *implies that* M_3 *is* MCM *if and only if* $\mathrm{Hom}_S(f, S)$ *is surjective.*

Maximal Cohen-Macaulay modules

Lemma 6.8 - part 2.

(*iii*) A module M is MCM over S if and only if $M^* = \mathrm{Hom}_S(M, S)$ is MCM as a right module over S^{op}. Furthermore, MCM's are reflexive, i.e. $M = M^{**}$, and a sequence of such is short exact in mod-S if and only if the dual sequence in mod-S^{op} is exact.
In other words, the functor $\mathrm{Hom}_S(-, S)$ induces an exact duality between MCM(S) and MCM(S^{op}).

(*iv*) Any module in mod-S admits a finite resolution by MCM's of lenght at most equal to vdim S. (In such a resolution all but the last module can chosen to be finitely generated projective.)

Maximal Cohen-Macaulay modules

(i) : It follows immediatly from the defintion, because

$$\mathrm{Ext}^i_S(P, S) = 0$$

for every P finitely generated projective.

(ii) : It follows easily from the long exact sequence induced by $\mathrm{Ext}^\bullet_S(-, S)$.

(iii) : From a projective resolution of M, by dualizing, we can get a projective co-resolution of M^* in mod-S^{op}, also using the definition of maximal Cohen-Macaulay. From this one can conclude the proof, but it's not trivial. In fact it is needed that for strongly Gorenstein rings, maximal Cohen-Macaulay modules are exactly the modules which are syzygy modules of arbitrarily high order. This last assertion is the reason why we focus on strongly Gorenstein rings. For this and (iv), see [Buchweitz, "Maximal Cohen-Macaulay Modules and Tate Cohomology over Gorenstein Rings", Lemma 4.2.2].
Furthermore, dualizing once again, it follows that M is reflexive.

The main theorem

Theorem 6.9.

Let S be a left and right noetherian ring, of finite injective dimension as a module over itself on either side. Then

(1) *The syzygy functor Ω_0, defined in Definition 5.16 induces an equivalence functor - denoted by the same symbol -*

$$\Omega_0 \colon \underline{\mathrm{APC}(S)} \to \underline{\mathrm{MCM}(S)}$$

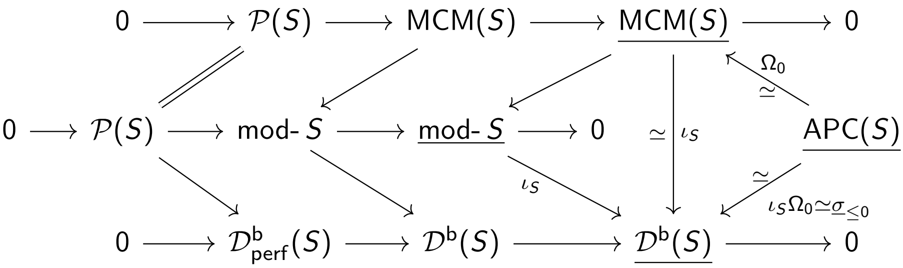
(2) *The restriction of ι_S, defined in Lemma 5.11, to $\underline{\mathrm{MCM}(S)}$ yields an equivalence of categories - again denoted by the same symbol -*

$$\iota_S \colon \underline{\mathrm{MCM}(S)} \to \underline{\mathcal{D}^{\mathrm{b}}(S)}$$

(3) *The triangulated structures induced on $\underline{\mathrm{MCM}(S)}$ by either Ω_0 or ι_S are the same, in the sense that the identity on $\underline{\mathrm{MCM}(S)}$ becomes a triangulated isomorphism of triangulated categories, and with respect to these structures, both functors Ω_0 and ι_S are triangulated functors of triangulated categories, transforming the corresponding translation functor T into an inverse of the loop space functor Ω_S restricted to $\underline{\mathrm{MCM}(S)}$.*

The main theorem

Before going into the proof, we resume the situation graphically. There is a diagram of categories and functors, whose rows are "exact sequences" of categories:



where the unlabeled morphisms are the canonical functors.

(*i*) Ω_0 takes its values in $\mathrm{MCM}(S)$:
Assume given an acyclic complex A of finitely generated projective S-modules. Then

$$\mathrm{Ext}^i_S(\Omega_0 A, S) = \mathrm{Ext}^{i+j}_S(\Omega_{-j} A, S)$$

for all $i > 0$, $j \geq 0$. Taking some $j > \mathrm{vdim}\, S - i$ we get that $\mathrm{Ext}^i_S(\Omega_0 A, S) = 0$. Since i was arbitrary > 0, we are done.

(*ii*) Ω_0 is surjective on objects:
Let M be a maximal Cohen-Macaulay module over S, $P(M^*) \to M^*$ a projective resolution of M^* in mod- S^{op}. As in the proof of the Lemma 6.8,

$$0 \to M^{**} \to \mathrm{Hom}_{S^{\mathrm{op}}}(P(M^*), S^{\mathrm{op}}) = P(M^*)^*$$

will be a projective co-resolution of M^{**} in mod- S. Now $M \simeq M^{**}$, as M is reflexive, and hence extending the projective co-resolution of M^{**} by a projective resolution of M yields a desired pre-image.

(*iii*) Ω_0 is a full functor:
Let $f\colon M \to N$ be a S-linear map of maximal Cohen-Macaulay modules over S. Extend it to a morphism $f^\bullet\colon P(M) \to P(N)$ between chosen projective resolutions and analogously $f^*\colon N^* \to M^*$ to a morphism $(f^*)^\bullet\colon P(N^*) \to P(M^*)$. Connecting $P(M)$ and $P(M^*)^*$ as well as $P(N)$ and $P(N^*)^*$ to have complete resolutions of M and N respectively, $f^\bullet$ and $((f^*)^\bullet)^*$ fit together to yield a morphism of these complete resolutions. By construction, this provides a pre-image of f. Hence Ω_0 is full.

To complete the proof of assertion (1), it remains to be seen that Ω_0 is faithful. Instead of proving this and then (2) we rather show that

$$\underline{\sigma_{\leq}}\colon \underline{\mathrm{APC}(S)} \to \underline{\mathcal{D}^{\mathrm{b}}(S)} \quad \text{is an equivalence of categories.}$$

The proof of the main theorem

This will imply the claim, since by Lemma 5.20, there are natural isomorphisms of functors

$$\underline{\sigma}_{\leq} \xrightarrow{\cong} \underline{\sigma}_{\leq 0} \xrightarrow{\cong} \iota_S \Omega_0$$

whence $\underline{\sigma}_{\leq}$ an equivalence gives that Ω_0 is faithful, and therefore also an equivalence by the above. And finally also ι_S will then be an equivalence of categories, establishing (2).

So we prove:

(iv) $\underline{\sigma}_{\leq}$ is essentially surjective:
In view of using Lemma 5.20, we find for any complex X in $\mathcal{D}^{\mathrm{b}}(S)$ an integer k, an object A in $\underline{\mathrm{APC}(S)}$ and a morphism $X \to \sigma_{\leq k} A$ of complexes whose mapping cone is perfect. Also, replacing if necessary X by one of its resolutions, we may assume that X itself is already a bounded above complex of finitely generated projective S-modules.

Then, all the syzygy modules of X which sit "far enough back" have to be maximal Cohen-Macaulay. More precisely, using the same argument as in (i) above,
$\sigma_{\leq k}(X) = \mathrm{Cok}(\mathrm{d}_X^k) = \mathrm{Im}(\mathrm{d}_X^{k+1} \colon X^k \to X^{k+1})$ is certainly MCM for $k \leq \min(i \ : \ \mathrm{H}^i(X) \neq 0) - \operatorname{vdim} S$, and $\sigma_{\leq k}(X)[k]$ is a resolution of this module which can be extended to an acyclic complex of projectives by the argument already used in (ii) above.

(v) $\underline{\sigma_{\leq}}$ is full and faithful:
We use Verdier's criterion, which can be found in Verdier, Catégories Derivées, Etat 0, in SGA 4 1/2, 262-311:
It suffices to prove that for a given perfect complex Y in $\mathcal{D}^{\mathrm{b}}(S)$ and an object A in $\underline{\mathrm{APC}(S)}$ there exists an integer k such that all morphisms in $\mathcal{D}^{\mathrm{b}}(S)$ from $\sigma_{\leq k}A$ into Y are zero.

The proof of the main theorem

As $\sigma_{\leq k}A$ is a bounded above complex of projective modules, morphisms in $\mathcal{D}^{\mathrm{b}}(S)$ from $\sigma_{\leq k}A$ into Y are in bijection with homotopy classes of (actual) morphisms of complexes and it is hence to show that any such morphism is indeed zero-homotopic.
Furthermore, it is enough to prove this assertion in case $Y = P[-i]$, P a finitely generated projective S-module and i an integer, as these objects generate - up to isomorphisms in $\mathcal{D}^{\mathrm{b}}(S)$ - any perfect complex by forming mapping cones.
Now, in this particular case, take any $k > i$ and let $f^\bullet$ be a complex morphism from $\sigma_{\leq k}A$ to $P[-i]$:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{i-1} & \xrightarrow{\mathrm{d}^i} & A^i & \xrightarrow{\mathrm{d}^{i+1}} & A^{i+1} & \longrightarrow & ?\ \cdots \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle f^i} & & \big\downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & P & \longrightarrow & 0 & \longrightarrow & 0\ \cdots
\end{array}$$

The proof of the main theorem

The S-linear map f^i factors hence necessarily over $\mathrm{Cok}(\mathrm{d}^i)$, ket $g\colon \mathrm{Cok}(\mathrm{d}^i) \to P$ be the induced map. It remains to show that g can be further factored through the inclusion (by choice of k) of $\mathrm{Cok}(\mathrm{d}^i) = \Omega_{-i}A$ into A^{i+1}.

But from the exact sequence

$$0 \to \Omega_{-i}A \to A^{i+1} \to \Omega_{-i-1}A \to 0$$

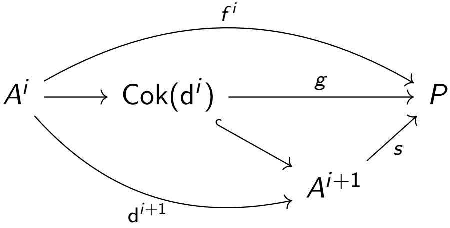
we have the long exact sequence

$$0 \to \mathrm{Hom}(\Omega_{-i-1}A, P) \to \mathrm{Hom}(A^{i+1}, P) \to \mathrm{Hom}(\Omega_{-i}A, P) \to \mathrm{Ext}^1_S(\Omega_{-i-1}A, P)$$

so it follows that the obstruction for that factorization lies in $\mathrm{Ext}^1_S(\Omega_{-i-1}A, P)$, which vanishes as P is finitely generated projective and $\Omega_{-i-1}A$ is still maximal Cohen-Macaulay by the argument in (1) above.

The proof of the main theorem

Thus we have the commutative diagram



Finally, (3) is just a reformulation of Lemma 5.11 and Lemma 5.18.

Bibliography

- "The Derived Category", Hongmiao Yu;
- "Triangulated Categories and Local Cohomology Functor", Martina Rovelli;
- "Maximal Cohen-Macaulay Modules and Tate Cohomology over Gorenstein Rings", Ragnar Olaf Buchweitz.